Ellipsoidal tube MPC of robots carrying glass plates

Xuhui Feng∗, Jiaqi C. Li, Mario E. Villanueva, Jürgen Pannek, Boris Houska

Abstract—This paper is about time-optimal robust model predictive control of a robot arm that carries a glass plate. We model constraints on the strains in the extremal fibers of the glass plate based on the section modulus and tensile strength to avoid breakages. In order to synthesize a control strategy, we propose to use a tailored ellipsoidal tube based model predictive control scheme, which can deal with these nonlinear strain constraints of the glass plate. The necessity of modeling the strains in the fibers as well as the properties of the proposed robust control method are illustrated in a realistic case study for a KUKA youBot model, which is simulated in the presence of process noise.

I. INTRODUCTION

With the rapid development of industrial robots and gradual expansion of scope of applications in recent years, especially in the area of cyberphysical systems [18] and human robot collaboration [19], the problem on reliable manipulation has become highly relevant. As robot motions are often optimized in order to minimize time or energy, even nominal solutions may lead to extreme or dangerous maneuvers. In practice, model uncertainties as well as external disturbances, such as joint frictions and varying contact points, have to be taken into account in the design of the controller to ensure safety. This is especially crucial if (i) humans are interacting with the robots [2] or if (ii) dangerous or fragile goods are transported by a robot [1].

The question how to model and analyze disturbances has been addressed rigorously in the context of PID and LQR control as well as Kalman filter based methods for estimating the system states [5], [16] to control basic movements. However, these control strategies work well for simple disturbance rejection, set point stabilization and simple tracking maneuvers only. The planning of more complex optimal control strategies, particularly in the presence of obstacles, changing environments, or challenging geometrical constraints, requires more advanced control tools. In this regard, adaptive control [9], [14], sliding mode control [22], [8], and disturbance observer design [7], [20] have shown to be suitable tools for planning complex control maneuvers in robotics. However, these methods are often based on linearization of the dynamics or various modeling assumptions, which neglect intrinsic properties of the model. Multi-link robotic manipulators are highly nonlinear and constrained systems. Thus a control scheme built on linearization or model simplification may be inapplicable.

Nonlinear model predictive control (MPC) [23] has shown its capability for handling robotic manipulators. Over the past 30 years, nonlinear MPC has grown mature and shown its capability of handling constrained processes by optimizing over manipulated inputs [10]. Although standard MPC controllers exhibit a certain inherent robustness in the sense that the controller is running in closed-loop mode [21], such controllers do not take uncertainties into account and, consequently, critical constraints cannot be guaranteed to be satisfied in the presence of model uncertainties or external disturbances. One extension of MPC dealing with worst-case scenarios in the presence of bounded uncertainty is termed robust MPC [15], [4]. Because, the focus of this paper is on a particular class of robust MPC controllers, named, Tube MPC controllers, we review only selected articles on robust MPC that are relevant in this context. A more general overview of robust MPC methods can be found in [13].

Tube MPC relies on the construction of robust forward invariant tubes (RFITs) [17], namely a tube enclosing all possible trajectories under a given feedback law. Its ellipsoidal variant is exploited in a recent paper [25] and corresponding implementation details are illustrated in [24], [11]. This main contribution of this paper is the application of the tube based MPC technique from [25] to a non-trivial case study that involves both a model of a 3DOF KUKA robot arm with three joints. Section II proposes a dynamic model for such a robust system. Here, a particular focus is on modelling a glass plate that is carried by the robot. This is important as the strains in the extremal fibers of the glass plate need to be bounded during highly dynamic operations in order to avoid that the glass breaks due to non-negligible inertial, centrifugal or Coriolis forces. Moreover, Section III elaborates on how these nonlinear constraints on the strains in the glass plate can be taken into account in a robust MPC design such that a safe pick and place maneuver can be obtained violating none of the constraints in the presence of uncertainties. In Section IV the corresponding developments are illustrated by simulating a realistic KUKA youBot robot arm. Section V concludes the paper.

II. A 3DOF ROBOT ARM AND GLASS PLATE MODEL

This section is divided into two parts. Section II-A outlines the dynamic model of a robot arm and Section II-B introduces a model of the glass plate that is carried by the robot.


A. Dynamic model of the robot

We consider a robot arm with three degrees of freedom as sketched in Fig. 1. The robot arm is composed of two links and one gripper carrying a glass plate. In the following,

\[ e_x = [1 \ 0 \ 0]^T, \ e_y = [0 \ 1 \ 0]^T, \ e_z = [0 \ 0 \ 1]^T \]

denote orthogonal basis vectors, where \( e_z \) is pointing to the sky. There are three joints, which can be used to place the centers of gravity of the links and gripper at

\[
R_1(\theta) = \frac{h_1}{2} e_z, \\
R_2(\theta) = h_1 e_x + \frac{l_1}{2} S_1(\theta_1) S_2(\theta_2) e_y, \\
R_3(\theta) = h_1 e_x + \left( l_1 + \frac{l_2}{2} \right) S_1(\theta_1) S_2(\theta_2) e_y,
\]

respectively. Here, \( h_1 \) and \( l_1 \) are the lengths of the links, \( l_2 \) the length of the gripper, and \( \theta = (\theta_1, \theta_2, \theta_3)^T \) stacks the angles of the joints. Moreover,

\[
S_1(\theta_1) = \begin{pmatrix} 
\cos \theta_1 & -\sin \theta_1 & 0 \\
\sin \theta_1 & \cos \theta_1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
S_2(\theta_2) = \begin{pmatrix} 
1 & 0 & 0 \\
0 & \cos \theta_2 & -\sin \theta_2 \\
0 & \sin \theta_2 & \cos \theta_2
\end{pmatrix}, \\
S_3(\theta_3) = \begin{pmatrix} 
\cos \theta_3 & 0 & \sin \theta_3 \\
0 & 1 & 0 \\
-\sin \theta_3 & 0 & \cos \theta_3
\end{pmatrix},
\]

denote the associated rotation matrices. The center of gravity of the glass plate is at

\[
\bar{R}(\theta) = R_3(\theta) + \frac{l_2 + c_3}{2} S_1(\theta_1) S_2(\theta_2) e_y.
\]

In order to work out the kinetic energy of the robot, we need to work out the angular velocities of all links, the gripper, and the glass plate. They are given by

\[
\omega_1(\dot{\theta}) = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \sin \theta_2 \end{pmatrix}, \quad \omega_2(\theta, \dot{\theta}) = \begin{pmatrix} \dot{\theta}_2 \\ \dot{\theta}_1 \cos \theta_2 \\ \dot{\theta}_3 + \dot{\theta}_1 \sin \theta_2 \end{pmatrix}, \quad \omega_3(\theta, \dot{\theta}) = \begin{pmatrix} \dot{\theta}_2 \cos \theta_3 - \dot{\theta}_1 \sin \theta_3 \cos \theta_2 \\ \dot{\theta}_3 + \dot{\theta}_1 \sin \theta_2 \\ \dot{\theta}_2 \sin \theta_3 - \dot{\theta}_1 \cos \theta_3 \sin \theta_2 \end{pmatrix}.
\]

and

\[
\omega_3(\theta, \dot{\theta}) = \bar{\omega}(\theta, \dot{\theta}) = \begin{pmatrix} \dot{\theta}_2 \cos \theta_3 - \dot{\theta}_1 \sin \theta_3 \cos \theta_2 \\ \dot{\theta}_3 + \dot{\theta}_1 \sin \theta_2 \\ \dot{\theta}_2 \sin \theta_3 - \dot{\theta}_1 \cos \theta_3 \sin \theta_2 \end{pmatrix}.
\]

The moment of inertia of the glass plate is assumed to be given by

\[
\bar{J} = \frac{\bar{m}}{12} \begin{pmatrix} c_1^2 + c_2^2 & 0 & 0 \\ 0 & c_2^2 + c_3^2 & 0 \\ 0 & 0 & c_1^2 + c_3^2 \end{pmatrix},
\]

where \( \bar{m} = \rho c_1 c_2 c_3 \) is the mass of the glass plate, \( \rho \) its density, and \( c_1, c_2 \) and \( c_3 \) the lengths of the edges of the plate as sketched in Figure 1. Now, the kinetic energy \( E_k \) of the robot and the glass plate is given by

\[
E_k = \frac{1}{2} \sum_{i=1}^{3} (m_i ||\dot{R}_i(\theta)||^2 + \omega_i(\theta, \dot{\theta})^T J \omega_i(\theta, \dot{\theta})) \\
+ \frac{1}{2} (\bar{m} ||\dot{\bar{R}}||^2 + \bar{\omega}(\theta, \dot{\theta})^T \bar{J} \bar{\omega}(\theta, \dot{\theta})).
\]

(1)

Here, \( m_i \) and \( J_i \) denote the mass and the moments of inertia of robot links as well as the gripper. Similarly, the potential energy is given by

\[
E_p(\theta) = m_1 g \frac{h_1}{2} + m_2 g (h_1 + \frac{l_1}{2} \sin \theta_2) \\
+ m_3 g (h_1 + l_1 \sin \theta_2 + \frac{l_2}{2} \sin \theta_2) \\
+ \bar{m} g (h_1 + l_1 \sin \theta_2 + l_2 \sin \theta_2),
\]

where \( g \) denotes the gravitational constant. The Lagrangian function is given by

\[
L(\theta, \dot{\theta}) = E_k(\theta, \dot{\theta}) - E_p(\theta),
\]

(2)

i.e., we have

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = u_i, \quad i = 1, 2, 3,
\]

(3)

where \( u_1, u_2 \) and \( u_3 \) are the torques at the three joints. Next, we introduce the shorthands \( \Gamma(\theta, \dot{\theta}) = \nabla_{\dot{\theta}} L(\theta, \dot{\theta}), \)

\[
h(\theta, \dot{\theta}) = \nabla_{\theta} L(\theta, \dot{\theta}) - \nabla_{\dot{\theta}} L(\theta, \dot{\theta}) \dot{\theta}, \quad u = [u_1, u_2, u_3]^T,
\]

such that (2) and (3) can be written in the form

\[
\ddot{\theta} = \Gamma(\theta, \dot{\theta})^{-1} (h(\theta, \dot{\theta}) + u).
\]

Under the additional assumption that the control \( u \) is Lebesgue-integrable and Lipschitz continuous with constant \( k_L \), \( u \) can be considered as an auxiliary state satisfying an ODE of the form \( \dot{u}(t) = v(t) \), where \( v : \mathbb{R} \rightarrow \mathbb{R}^3 \) is the new control variable bounded by \( -k_L \leq v_i(t) \leq k_L, i \in \{1, \ldots, 3\} \), on the time horizon \([0, T]\). In the following, we stack all differential states into one vector
\[ \mathbf{s} = [\theta^T, \dot{\theta}^T, u^T]^T. \] Moreover, the differential equations for all three angles may be perturbed by external disturbances \( \mathbf{w} = [w_1, w_2, w_3]^T. \) The corresponding perturbed dynamic equations can be summarized as a first order differential equation system in standard form,

\[ \dot{\mathbf{s}}(t) = f(\mathbf{s}(t), \mathbf{w}(t)) + G\mathbf{v}(t), \]  

with right hand function

\[ f(s, w) = [\dot{\theta} + w, \Gamma(s)^{-1}(h(s) + u)]^T \]

and \( G = [0^6 \times 3, I_3]^T. \)

### B. Glass plate model

In order to ensure that the glass plate does not break during a maneuver, one has to model the strains that act at the extremal fibers. The following considerations specialize on the case that the glass plate is grasped by a simple gripper, but the corresponding derivations can also be generalized for other mechanical solutions, e.g., if vacuum cups are used. Let us introduce the shorthand

\[ \hat{R}(\theta, r) = R_3(\theta) + S_{123}(\theta) \left( \begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \right) \]

for the position of an infinitesimal mass point in the glass plate at distance \( r \in \mathbb{R}^3 \) from the gripper, where \( S_{123}(\theta) = S_1(\theta_1)S_2(\theta_2)S_3(\theta_3) \). Moreover, let

\[
\begin{align*}
    a(s, r) &= \frac{\partial^2 \hat{R}(\theta, r)}{\partial \theta^2} r e_x + g e_z \\
    &= \frac{\partial^2 \hat{R}(\theta, r)}{\partial \theta^2} \frac{\partial \theta}{\partial \theta}^2 + g e_z \\
    &\quad + \frac{\partial}{\partial \theta} \hat{R}(\theta, r) \right( \Gamma(\theta, \dot{\theta})^{-1} (h(\theta, \dot{\theta}) + u) \)
\end{align*}
\]

denote the acceleration of this mass point that is due to graviation. Therefore, the torque and strain force of the glass plate at the gripper are given respectively by

\[
\begin{align*}
    M(s) &= \rho \int_0^{c_1} \int_{\mathbb{R}^3} \hat{R}(\theta, r) \times a(s, r) \, dr_1 \, dr_2 \, dr_3 \\
    F(s) &= \rho \int_0^{c_1} \int_{\mathbb{R}^3} a(s, r) \, dr_1 \, dr_2 \, dr_3.
\end{align*}
\]

Here, \( \times \) denotes the outer product between two vectors in \( \mathbb{R}^3 \). Notice that the above integrals can be worked out explicitly by using state of the art computer algebra tools, but the corresponding expressions are rather long and therefore not displayed as part of this paper.

In order to ensure that the glass plate does not break at the gripper, we need to bound the strain of the external fiber at the rectangular area at the front side of the gripper. If \( c_e \) denotes the length of this front edge of the gripper (see Figure 1), the associated rectangular area and section modulus are given by

\[ A_1 = c_e c_2 \quad \text{and} \quad S_1 = \frac{c_e c_2^2}{6}. \]

Thus, we must ensure that

\[ \sigma_x(s) = \frac{|M(s)^T S_{123}(\theta)^T e_x|}{S_1} + \frac{|F(s)^T S_{123}(\theta)^T e_y|}{A_1} \leq \tau, \]

where \( \tau \) denotes the tensile strength. Similarly, we must ensure that

\[ \sigma_y(s) = \frac{|M(s)^T S_{123}(\theta)^T e_y|}{S_2} + \frac{|F(s)^T S_{123}(\theta)^T e_x|}{A_2} \leq \tau, \]

with \( A_2 = c_y c_2 \) and \( S_2 = \frac{c_e c_2^2}{b} \) (see [3] for details on section modulus and tensile strength). Here, \( c_e \ll c_1 \) denotes the length of the other edge of the gripper.

### III. ROBUST TIME-OPTIMAL MPC

#### A. Time-Optimal MPC

Nominal time-optimal maneuvers can be found by solving optimal control problems of the form

\[
\min_{s, v, t} \quad T
\]

subject to

\[
\begin{align*}
    \dot{s}(t) &= f(s(t), 0) + Gv(t), \quad 0 \leq t \leq T \\
    s(0) &= s_0, \quad s(T) = s_T \\
    \theta &\leq \theta(t) \leq \bar{\theta}, \quad u \leq u(t) \leq \bar{u}, \\
    -k_L \leq v(t) \leq k_L, \\
    \text{constraints (5) and (6),}
\end{align*}
\]

where \( s_0 \) denotes the initial state, \( s_T \) the target state, \( \theta \) as well as \( \bar{\theta} \) the lower and upper bounds of the angles of the joints, and \( u \) as well as \( \bar{u} \) the lower and upper bounds of the torque \( u \). Notice that this optimal control problem is based on the assumption that there is no uncertainty, \( w = 0 \). However, in the presence of process noise, it is advisable to use a closed-loop controller instead, which can account for uncertainties explicitly. Our goal is to design control actions, which ensure that the strain constraints on the glass plate are satisfied for a whole set of uncertainty scenarios, as elaborated in the section below.

#### B. Tube based Time-Optimal MPC

The differential equation (4) is perturbed by the process noise \( w \). In the following, we assume that there exists a pair \((q_w, Q_w)\) \( \in \mathbb{R}^{n_w} \times \mathbb{S}^{n_w} \) such that

\[
\mathbb{W} = \mathcal{E}(q_w, Q_w) = \{ q_w + Q_w^{1/2} \nu \mid \nu^T \nu \leq 1 \}.
\]

The set-valued function \( X : [0, T] \times \mathbb{R}^{n_z} \times \mathbb{L}^2_{n_u} \to \Pi(\mathbb{R}^{n_z}) \) is called an RFT for (4) if there exists a feedback law \( \mu : [0, T] \times \mathbb{R}^{n_z} \to \mathcal{V} = [-k_L, k_L]^3 \) such that any solution of the controlled system

\[
\begin{align*}
    \forall t \in \mathbb{R} : \quad \dot{s}(t) &= f(s(t), w(t)) + G\mu(t, x(t)), \\
    s(0) &= s_0
\end{align*}
\]

with \( s(t) \in X(t, s_0, \mu) \) satisfies \( s(t') \in X(t', s_0, \mu) \) for all \( t, t' \in [0, T] \). Here, \( \mathbb{L}^2_{n_u} \) denotes the set of square integrable functions mapping from \( \mathbb{R}^{n_u} \) to \( \mathbb{R} \) and \( \Pi(\mathbb{R}^{n_z}) \) the power set of \( \mathbb{R}^{n_z} \). Now, the goal of the following considerations is
to find a computationally tractable approximation of the tube
MPC problem
\[
\inf_{T, \mu : R \times R^{nx} \to V} T
\]
\[
\text{s.t. } \{X(t, s_0, \mu) \subseteq X, \ldots \} \quad \text{loop simulation based on the robust MPC controller (14). All}
\]
numerical results are found by using ACADO Toolkit [12].

In [25], where the following theorem is established.

C. Ellipsoidal Parameterization

In order to approximately solve (9), we follow an ellipsoidal parameterization approach that has been proposed in [25], where the following theorem is established.

**Theorem 1:** Let \( v_{\text{ref}} : [0, T] \to V \) and \( \lambda : [0, T] \to R_+ \), \( \kappa : [0, T] \to R_+ \), and \( K : [0, T] \to R^{3 \times 9} \) be any given integrable functions and let \( q \) and \( Q \) be solutions of the ODE
\[
\dot{q}(t) = f(q(t), 0) + Gv_{\text{ref}}(t),
\]
\[
\dot{Q}(t) = A(t)Q(t) + Q(t)A(t)^T + \lambda(t)Q(t) + \frac{1}{\kappa(t)}B(t)QwB(t)^T
\]
\[
+ \kappa(t)Q(t) + GK(t)Q(t) + Q(t)K(t)^TG^T
\]
\[
q(0) = s_0, \quad Q(0) = 0
\]
(10)
with
\[
A(t) = \frac{\partial f}{\partial s}(q(t), 0) \quad \text{and} \quad B(t) = \frac{\partial f}{\partial w}(q(t), 0).
\]

Now, if the function \( \Omega_n : R \to S^d_+ \) in the above differential equation satisfies
\[
f(q(t), q_u(t)) + A(t)(s(t) - q(t)) + B(t)(w(t) - q_u(t)) \in E(0, \Omega_n(t)),
\]
and if
\[
v_{\text{ref}, i}(t) + \sqrt{K_i(t)Q(t)K_i(t)^T} \leq k_L,
\]
\[
v_{\text{ref}, i}(t) - \sqrt{K_i(t)Q(t)K_i(t)^T} \geq -k_L,
\]
for \( i \in \{1, 2, 3\} \) with \( K_i \) denoting the \( i \)-th row of the matrix \( K \), then the set \( X(t) = E(q(t), Q(t)) \) is a robust forward invariant tube.

In order to arrive at a practical implementation, we need to work out the nonlinearity bounder \( \Omega_n \) needed in Theorem 1. One way of constructing such a bounder satisfying the conditions in this theorem, has been introduced in [24], where it is suggested to use Hessian matrix bounds. The corresponding computational is outlined in Algorithm 1.

Next, all state trajectories in the tube \( E(q(t), Q(t)) \) satisfy the state constraints (5) and (6) if
\[
|\sigma_x(q) + \sqrt{\frac{\partial \sigma_y^T}{\partial s}(q)}\frac{\partial \sigma_x}{\partial s}(q) \pm \Omega_{\sigma_x}(q)| \leq \tau
\]
(12)

**Algorithm 1: Computation of the Bounder of Nonlinearity Estimate.**

**Input:** Central path \( q(t) \) and shape matrix \( Q(t) \).

**Begin:**
1) Compute the Hessian approximation of the components of the central path,
\[
H_1(t) = (H_1(t))^{2-1/2} (H_1(t))^{1/2} H_1(t)^{-1}.
\]
2) Compute the weighting matrix,
\[
W(t) = \left( H_1(t)^{-1} (H_1(t))^{2-1/2} (H_1(t))^{1/2} H_1(t)^{-1}\right)^{1/2}.
\]
3) Compute an interval bound on the Frobenius norm of the weighted Hessian matrix:
\[
\|H_1(t)W(t)||F| = \left\{ \xi \in E(q(t), Q(t)) \right\},
\]
where
\[
H_1^0(t) \geq \left\{ \frac{\partial \sigma_y}{\partial s}(q) \right\}_{\xi \in E(0, Q_w)}.
\]
4) Compute the shape matrix of the bounder of nonlinearity estimate,
\[
\Omega_{\sigma}(t) = \frac{1}{4} \text{diag} (\gamma^2(t)||W(t)||F|Q(t)||^2).
\]
**Output:** \( \Omega_{\sigma}(t) \).

and
\[
|\sigma_y(q) + \sqrt{\frac{\partial \sigma_y^T}{\partial s}(q)}\frac{\partial \sigma_y}{\partial s}(q) \pm \Omega_{\sigma_y}(q)| \leq \tau,
\]
(13)

where \( \Omega_{\sigma_y}(q) \) and \( \Omega_{\sigma_y}(q) \) denote the nonlinearity estimates of the strain functions \( \sigma_x \) and \( \sigma_y \). Notice that these functions can be computed analogous to \( \Omega_n \) in Algorithm 1. Thus, a conservative reformulation of (9) is given by the following optimal control problem:

\[
\inf_{q, Q, v_{\text{ref}}, \lambda, \kappa, K, T} T
\]
\[
\text{s.t. } \text{ODEs (10),}
\]
\[
q(0) = s_0, Q(0) = I,
\]
\[
q(T) = s_T, Q(T) = Q_T,
\]
\[
\text{additional constraints (11), (12), and (13),}
\]
\[
\lambda(t), \kappa(t) > 0, Q(t) \in S^d_+, \forall t \in [0, T],
\]
\[
k_L \leq v_{\text{ref}, i}(t) \leq k_L, \quad \forall t \in [0, T],
\]
(14)

where \( X_T = E(s_T, Q_T) \) models an ellipsoidal terminal region. More details about the theoretical properties of this reformulation as well as an initialization scheme can be found in [11].

**IV. CASE STUDY**

This section presents numerical results for the nominal time-optimal control problem (7) as well as a closed loop simulation based on the robust MPC controller (14). All numerical results are found by using ACADO Toolkit [12].
with multiple-shooting and a piecewise constant control discretization with 10 pieces. The nonlinearity estimate is computed offline by using Algorithm 1. The required bounds have all been found by using the software library MC++ [6]. All model parameters of robot and glass plate can be found in Table I. In addition, the inertia of the three links are given by

\[
J_1 = \text{diag}(29.525, 60.091, 58.821) \quad \text{[kg \cdot cm}^2\text{]}, \\
J_2 = \text{diag}(31.145, 5.483, 31.631) \quad \text{[kg \cdot cm}^2\text{]}, \\
\text{and} \quad J_3 = \text{diag}(1.934, 1.602, 0.689) \quad \text{[kg \cdot cm}^2\text{]}.
\]

The shape matrix of the uncertainty model is given by

\[
Q_w = \text{diag} \left( 0.5^2, 0.5^2, 0.5^2 \right) \quad \left[ \frac{\text{rad}^2}{\text{s}^2} \right].
\]

A. Nominal Time-Optimal Control

The red dashed line in Figure 3 shows the results for the strains \(\sigma_x\) and \(\sigma_y\) in the glass of the nominal time-optimal control problem (7) as a function of time. In addition, the solid line shows the associated closed-loop simulation result of the whole system with process noise. As we can see in Figure 3, one of the constraint on the maximum strain in the extremal fibers of the glass plate is violated if uncertainty is present. Notice that the minimum time of the nominal maneuver is \(T^* \approx 0.2135\text{s}\).

B. Robust Time-Optimal MPC

Figure 2 shows the trajectory (solid red line) of the robot arm for the robust maneuver in 2D and in 3D, i.e., the maneuver that is obtained by running the robust closed loop model predictive controller (14). In addition, the plot

Fig. 2. The trajectory of the robot arm and the orientation of the glass plate at different time instances in all 2D projections and in 3D for the same randomly generated disturbances.

Fig. 3. Strains in the glass plate for a nominal time-optimal control maneuver. The black dashed line denotes the tensile strength of the glass plate.

Fig. 4. The uncertainty region for the extremal strains in the glass plate. The black dashed line denotes the tensile strength of the glass plate.
TABLE I
PHYSICAL PARAMETERS AND STATE CONSTRAINT BOUNDS.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acceleration of gravity</td>
<td>$g$</td>
<td>9.81 $\text{m/s}^2$</td>
</tr>
<tr>
<td>Mass of three links</td>
<td>$m_{1,2,3}$</td>
<td>1.39, 1.318, 0.687 $\text{kg}$</td>
</tr>
<tr>
<td>Lengths of links</td>
<td>$l_{1,2}$</td>
<td>0.155, 0.135 $\text{m}$</td>
</tr>
<tr>
<td>Height of Joint 2</td>
<td>$h_2$</td>
<td>0.147 $\text{m}$</td>
</tr>
<tr>
<td>Size of the gripper</td>
<td>$c_{g1}$, $c_{g2}$</td>
<td>0.02, 0.008 $\text{m}$</td>
</tr>
<tr>
<td>Lipschitz constant</td>
<td>$k_L$</td>
<td>20 $\text{N/m}$</td>
</tr>
<tr>
<td>Size of the glass plate</td>
<td>$c_{1,2,3}$</td>
<td>0.7, 0.1, 0.002 $\text{m}$</td>
</tr>
<tr>
<td>Tensile strength</td>
<td>$\tau$</td>
<td>45 $\text{MPa}$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$2.5 \times 10^3 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>Bounds on Joint 1</td>
<td>$[\theta_1, \bar{\theta}_1]$</td>
<td>$[-2.95, 2.95] \text{ rad}$</td>
</tr>
<tr>
<td>Bounds on Joint 2</td>
<td>$[\theta_2, \bar{\theta}_2]$</td>
<td>$[-1.13, 1.57] \text{ rad}$</td>
</tr>
<tr>
<td>Bounds on Joint 3</td>
<td>$[\theta_3, \bar{\theta}_3]$</td>
<td>$[-2.88, 2.88] \text{ rad}$</td>
</tr>
<tr>
<td>Bounds on $u_1$</td>
<td>$[\bar{u}_1, \bar{u}_1]$</td>
<td>$[-10, 10] \text{ N} \cdot \text{m}$</td>
</tr>
<tr>
<td>Bounds on $u_2$</td>
<td>$[\bar{u}_2, \bar{u}_2]$</td>
<td>$[-10, 10] \text{ N} \cdot \text{m}$</td>
</tr>
<tr>
<td>Bounds on $u_3$</td>
<td>$[\bar{u}_3, \bar{u}_3]$</td>
<td>$[-10, 10] \text{ N} \cdot \text{m}$</td>
</tr>
<tr>
<td>Initial position</td>
<td>$\theta(0)$</td>
<td>$[0.0, 0.5]^T \text{ rad}$</td>
</tr>
<tr>
<td>Initial angular velocity</td>
<td>$\dot{\theta}(0)$</td>
<td>$[0, 0, 0]^T \text{ rad/s}$</td>
</tr>
<tr>
<td>Target position</td>
<td>$\theta(T)$</td>
<td>$[\frac{\pi}{2}, 0, 0]^T \text{ rad}$</td>
</tr>
<tr>
<td>Target angular velocity</td>
<td>$\dot{\theta}(T)$</td>
<td>$[0, 0, 0]^T \text{ rad/s}$</td>
</tr>
</tbody>
</table>

visualizes the orientation of the glass plate at different time instances. Notice that this maneuver is robust with respect to uncertainties, but the time needed to perform the maneuver is now increased, $T^* = 0.2346$s. Next, Figure 4 shows the strains in the glass plate during the robust maneuver as a function of time visualizing the uncertainty tube around the nominal strain functions. Notice that all constraints are satisfied for all uncertainty scenarios ensuring a safe operation without breaking the glass.

V. CONCLUSION

This paper has presented a dynamic model for a robot arm carrying a fragile glass plate with a particular emphasis on modeling the strains in the extremal fibers of the plate close to the gripper of the robot arm. Equations (12) and (13) summarize our model for the bounds on the strains, which can be considered as additional constraints on the dynamic motion of the robot. The developments in Section III have focused on how these constraints can be satisfied for all uncertainty scenarios during a time-optimal pick-and-place maneuver by using a tailored robust tube-based MPC scheme. Our numerical results show how the proposed scheme can be used to design robust time-optimal motions in closed-loop mode.

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