HW10_ Solution

<u>Pr 8.14</u> (a) The sampler is linear: if $x_i(t)$, $i = 1, 2$, are the inputs then $x_i(nT_s)$, $i = 1, 2$ are the outputs. When the input is $ax_1(t) + bx_2(t)$, a combination of the previous two inputs, then the output will be $ax_1(nT_s) + bx_2(nT_s)$ or the combination of the previous outputs.
(b) For $T_s = 1$, the sampling of $x(t) = \cos(0.5\pi t)u(t)$ gives the signal

$$y(nT_s) = \cos(0.5\pi n)u[n]$$

while for $T_s = 1$ the sampling of $x(t - 0.5) = \cos(0.5\pi(t - 0.5))u(t - 0.5)$ gives

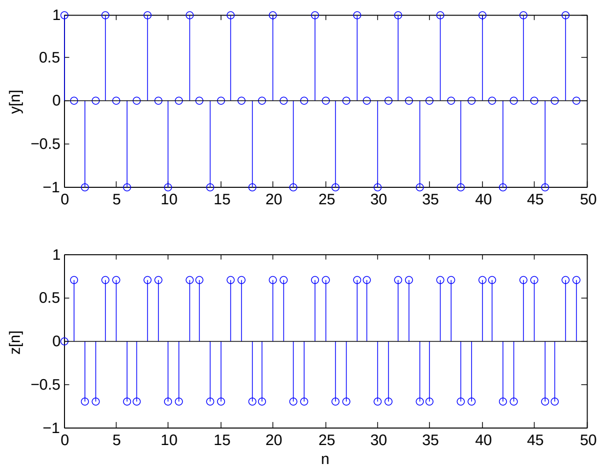
$$z(nT_s) = \cos(0.5\pi(n - 0.5))u(n - 0.5)$$

Let us compare the first five values

$$\begin{array}{ll}
y(0) = 1 & z(0) = 0 \\
y(1) = 0 & z(1) = \cos(0.25\pi) = 0.707 \\
y(2) = -1 & z(2) = \cos(0.75\pi) = -0.707 \\
y(3) = 0 & z(3) = \cos(1.25\pi) = -0.707 \\
y(4) = 1 & z(4) = \cos(1.75\pi) = 0.707 \\
\vdots &
\end{array}$$

showing clearly that $z(nT_s)$ is not $y(nT_s)$ shifted. Thus the sampler is time-varying. Notice that if we sample the analog signal shifted by exactly one or more sampling periods, the output signal is a shifted version of the originally sampled signal. But if the shift is not a multiple of the sampling period, it is not. To verify that the sampler is not TI we use the following script to plot $y[n]$ and $z[n]$ for $0 \leq n \leq 49$.

```
%%% Pr 8.14
clear all;clf
n=0:49;
y=cos(0.5*pi*n);
n1=1:49;
z=[0 cos(0.5*pi*(n1-0.5))];
figure(1)
subplot(211)
stem(n,y); ylabel('y[n]')
subplot(212)
stem(n,z); ylabel('z[n]'); xlabel('n')
```


y[n]
z[n]
n
1
0.5
0
−0.5
−1
0
5
10
15
20
25
30
35
40
45
50

Pr 8.15 (a) Yes, the quantizer is time-invariant. Indeed, $x(nT_s)$ and $x(nT_s - MT_s)$ will give the same values just shifted in time. In this case the M must be an integer as the sampled signal is discrete in time.
(b) The samples are given by $x(nT_s) = nT_s = 0.1n$. For $\Delta = 0.25$ the sample $x[1] = 0.1$ when quantized gives $\hat{x}[1] = 0$ since $0 < 0.1 < \Delta$. If we multiply the signal by 3 so that the value $3x[1] = 0.3$ would give $\hat{x}[1] = 1$ since $\Delta < 0.3 < 2\Delta$. Since the second output is not the first output multiplied by 3, the quantizer is not linear.
We saw before that the sampler is linear but time-varying, while the quantizer is time-invariant but non-linear, so the A/D converter which is composed of these systems is not LTI.

Pr 8.16 (a) If $\mathrm{w}[n] = u[n] - u[n-N]$ is the rectangular window, windowing the signal $x[n]$ gives $y[n] = x[n]\mathrm{w}[n]$. If we have the following input/output pairs

$$x_1[n] \quad \Rightarrow \quad y_1[n] = \mathrm{w}[n]x_1[n]$$
$$x_2[n] \quad \Rightarrow \quad y_2[n] = \mathrm{w}[n]x_2[n]$$

then

$$ax_1[n] + bx_2[n] \quad \Rightarrow \quad \mathrm{w}[n][ax_1[n] + bx_2[n]] = ay_1[n] + by_2[n]$$

thus the windowing process is linear.

(b)-(c) However, windowing it is not time invariant. If we window $x[n] = nu[n]$ using a rectangular window $\mathrm{w}[n] = u[n] - u[n-6]$ results in $y[n] = n(u[n] - u[n-6])$. If we shift $x[n]$ by 6 to get $x[n-6] = [n-6]u[n-6]$ (which is zero for $0 \leq n \leq 5$) then $x[n-6]\mathrm{w}[n]$ is zero for all n and not equal to $y[n-6]$ which is what we should get if windowing was time-invariant. The following script displays this.

```
%%% Pr 8.16
clear all; clf
w=[ones(1,6) zeros(1,20-6)];
n=0:19;
x=n;
x1=[zeros(1,6) x(1:20-6)];
figure(1)
subplot(211)
stem(n,w.*x);axis([0 19 0 5]);grid
subplot(212)
stem(w.*x1);axis([0 19 0 5]);grid
```

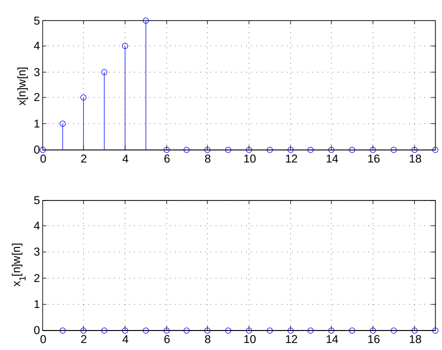


Figure 8.5: .

Pr 8.17 (a) The impulse response can be found solving the difference equation recursively, i.e., letting the output be $y[n] = h[n]$ and the input be $x[n] = \delta[n]$ and assuming zero initial conditions, i.e., $h[n] = 0$ for $n < 0$. We have then

$$h[n] = 0.15h[n-2] + \delta[n]$$

which for $n \geq 0$ gives

$$\begin{aligned}
h[0] &= 0.15h[-2] + 1 = 1 \\
h[1] &= 0.15h[-1] + 0 = 0 \\
h[2] &= 0.15h[0] + 0 = 0.15 \\
h[3] &= 0.15h[1] + 0 = 0 \\
h[4] &= 0.15h[2] + 0 = 0.15^2 \\
h[5] &= 0.15h[3] + 0 = 0 \\
&\vdots
\end{aligned}$$

or

$$h[n] = \begin{cases} 0.15^{n/2} & \text{for } n \geq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases}$$

(b) Letting the initial conditions be zero, the input $x[n] = \delta[n]$ and $y[n] = h[n]$ we have

$$h[n] = 0.15h[n-2] + \delta[n]$$

replacing $h[n-2] = 0.15h[n-4] + \delta[n-2]$ according to the equation for the difference equation, we get

$$h[n] = 0.15(0.15h[n-4] + \delta[n-2]) + \delta[n] = 0.15^2 h[n-4] + 0.15\delta[n-2] + \delta[n]$$

and again replacing $h[n-4] = 0.15h[n-6] + \delta[n-4]$ we have

$$\begin{aligned}
h[n] &= 0.15^2(0.15h[n-6] + \delta[n-4]) + 0.15\delta[n-2] + \delta[n] \\
&= 0.15^3 h[n-6] + 0.15^2\delta[n-4] + 0.15\delta[n-2] + \delta[n]
\end{aligned}$$

Repeating this process we realize that after more iterations we get that,

$$h[n] = \delta[n] + 0.15\delta[n-2] + 0.15^2\delta[n-4] + 0.15^3\delta[n-6] + \cdots$$

which is the same result as in the previous part.

(c) The following script using *filter* computes and plots $h[n]$

```
%%% Pr 8.17
clear all;clf
a=[1 0 -0.15];
b=1;
x=[1 zeros(1,29)];
h=filter(b,a,x);
n=0:29;
figure(1)
stem(n,h); axis([0 29 0 1]); grid;ylabel('h[n]'); xlabel('n')
```

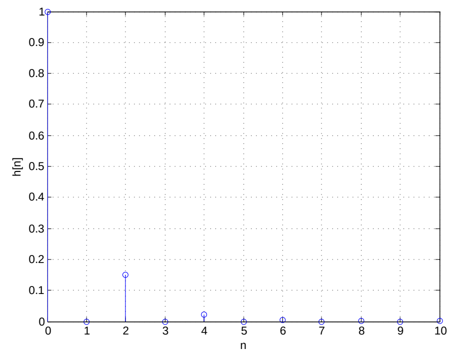


Figure 8.6: .

Pr 8.22 (a) The impulse response of the IIR filter is found by solving the difference equation

$$h_1[n] = \delta[n] - 0.5h_1[n-1] \qquad n \geq 0$$

obtained by letting $y_1[n] = h_1[n]$, $x[n] = \delta[n]$ and initial conditions equal to zero. Recursively we obtain

$$\begin{aligned} h_1[0] &= 1 \\ h_1[1] &= -0.5h_1[0] = -0.5 \\ h_1[2] &= -0.5h_1[1] = (-0.5)^2 \\ &\cdots \end{aligned}$$

which gives in general $h_1[n] = (-0.5)^n u[n]$.
The impulse response of the FIR system is obtained by letting $x[n] = \delta[n]$ and $y_2[n] = h_2[n]$ so that

$$h_2[n] = \delta[n] + 0.5\delta[n-1] + 3\delta[n-2] + \delta[n-5]$$

(b)(c) The condition for BIBO stability of a LTI system is that its impulse response be absolutely summable. The given IIR and FIR systems are LTI. The stability condition for the IIR system is that the following sum converges

$$\sum_{n=0}^{\infty} |h_1[n]| < \infty$$

It can be shown that this sum converges, i.e.,

$$\sum_{n=0}^{\infty} |(-0.5)^n| = \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2$$

therefore the IIR system is BIBO stable.
For the FIR, the stability condition is satisfied since we are adding a finite number of values. Indeed,

$$\sum_{n=0}^{\infty} |h_2[n]| = 1 + 0.5 + 3 + 1 < \infty$$

thus the FIR is BIBO stable.
(d) Because of the finite support of the impulse response of an FIR filter it is always absolutely summable and therefore FIR filters are always BIBO stable.