Signals and Systems Homework 4

March 15, 2017

Problem 1

Determine the Laplace transform of each of the following signals and specify the corresponding regions of convergence.

(1) $f(t) = e^{-a|t|}$

(2) $f(t) = |t|e^{-a|t|}$

(3) $f(t) = \sin(w|t|)$

(4) $f(t) = \cos(w|t|)$

(5) $f(t) = e^{-a|t|}\cos(w|t|)$

(6) $f(t) = e^{-a|t|+jw|t|}$

(7) $f(t) = |t|\sin(w|t|)$

(8) $f(t) = |t|^n e^{-a|t|}$, n is positive integer.

Solution

(1) $\mathcal{L}(s) = \frac{1}{a-s} + \frac{1}{a+s}$ ROC:$-a < \mathcal{R}e[s] < a$
(2) $\mathcal{L}(s) = \frac{1}{(a+s)^2} + \frac{1}{(a-s)^2}$ ROC:$-a < \mathcal{R}e[s] < a$
(3) it doesn't converges
(4) it doesn't converges
(5) $\mathcal{L}(s) = \frac{s+a}{(a+s)^2+w^2} + \frac{a-s}{(a-s)^2+w^2}$ ROC:$-a < \mathcal{R}e[s] < a$
(6) $\mathcal{L}(s) = j[\frac{w}{(a+s)^2+w^2} + \frac{w}{(a-s)^2+w^2}] + \frac{s+a}{(a+s)^2+w^2} + \frac{a-s}{(a-s)^2+w^2}$ ROC:$-a < \mathcal{R}e[s] < a$
(7) it doesn't converges
(8) $\mathcal{L}(s) = \frac{n!}{(a+s)^{n+1}} + \frac{n!}{(a-s)^{n+1}}$ ROC:$-a < \mathcal{R}e[s] < a$

Problem 2

Determine the Laplace transform of each of the following signals and specify the corresponding regions of convergence ,then plot the poles and zeros.

(1) $x(t) = e^{-4|t|} + e^{-5t}\sin(5t)u(t)$

(2) $x(t) = \int_{-\infty}^{t}(e^{-2\tau} - 1)u(\tau)d\tau + e^{-3t}u(t)$

(3)

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 3 \end{cases}$$

Solution

(1) Using an approach similar to Problem 1, we have

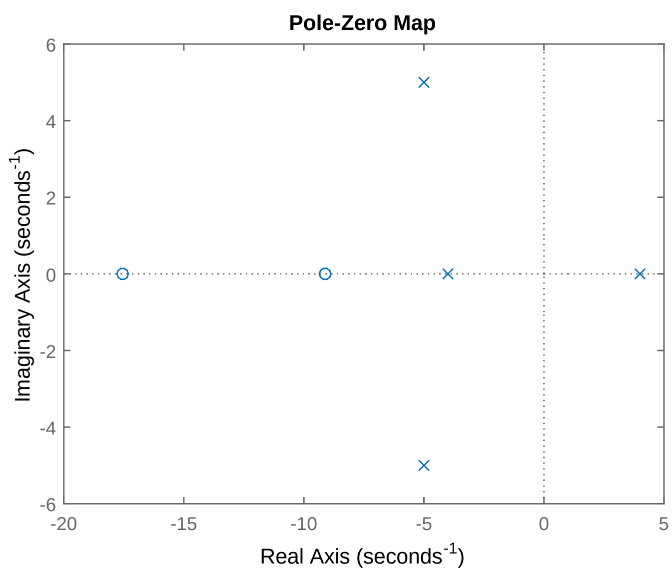
$$e^{-4|t|} \xrightarrow{\mathcal{L}} \frac{1}{4-s} + \frac{1}{4+s} \quad \text{ROC} : -4 < \mathcal{R}e[s] < 4$$

and

$$e^{-5t}\sin(5t)u(t) \xrightarrow{\mathcal{L}} \frac{5}{(s+5)^2+25} \quad \text{ROC} : \mathcal{R}e[s] > -5$$

Thus

$$X(s) = \frac{1}{4-s} + \frac{1}{4+s} + \frac{5}{(s+5)^2+25} \quad \text{ROC} : -4 < \mathcal{R}e[s] < 4.$$



(2) We can obtain the following expressions easily:

$$(e^{-2t} - 1)u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+2} - \frac{1}{s} \quad \text{ROC} : \mathcal{R}e[s] > 0.$$

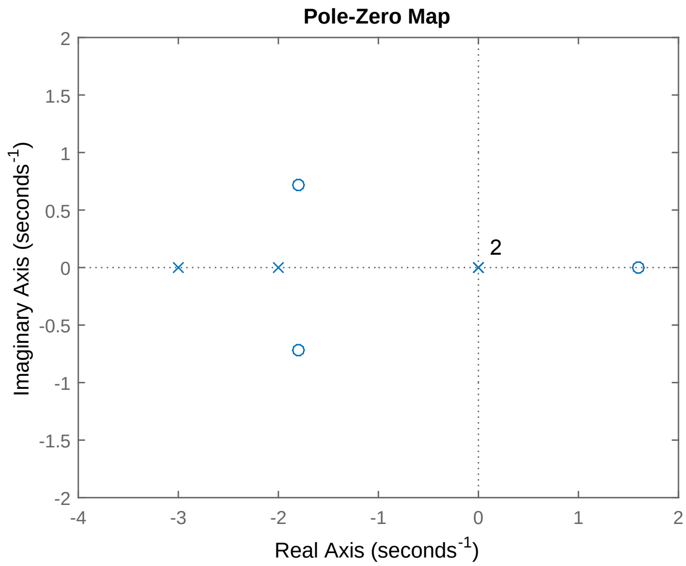
$$\int_{-\infty}^{t}(e^{-2\tau} - 1)u(\tau)d\tau \xrightarrow{\mathcal{L}} \frac{\frac{1}{s+2} - \frac{1}{s}}{s} \quad \text{ROC} : \mathcal{R}e[s] > 0,$$

and

$$e^{-3t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s+3} \quad \text{ROC} : \mathcal{R}e[s] > -3.$$

Therefore,

$$X(s) = \frac{-2}{(s+2)s^2} + \frac{1}{s+3} \quad \text{ROC} : \mathcal{R}e[s] > 0.$$



(3) It is not easy to obtain the Laplace transform of $x(t)$ easily, so we use the property of Laplace transform: $\frac{d}{dt}x(t) = sX(s)$. Then the second derivative of $x(t)$ is

$$\frac{d^2[x(t) - u(t-3)]}{dt^2} = \delta(t) + \delta(t-3) - 2\delta(t-1),$$

and the corresponding Laplace transform is

$$s^2[X(s) - \frac{e^{-3s}}{s}] = 1 + e^{-3s} - 2e^{-s}.$$

Therefore,

$$X(s) = \frac{1 + e^{-3s} - 2e^{-s}}{s^2} + \frac{e^{-3s}}{s}.$$

One zeros, $x \approx -0.422$.

Problem 3

A system is described by the following differential equation (see below). Find the expression for the transfer function of the system, Y (s)/X(s), assuming zero initial conditions.

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = \frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x$$

Consider what if the initial conditions are not zero.

Solution

(a) If the initial condition is zero, then

$$\mathcal{L}(\frac{d^n y(t)}{dt^n}) = s^n Y(s)$$

We have

$$\begin{aligned} s^3Y(s) + 3s^2Y(s) + sY(s) + Y(s) &= s^3X(s) + 4s^2X(s) + 6xX(s) + 8X(s) \\ Y(s)[s^3 + 3s^2 + s + 1] &= X(s)(s^3 + 4s^2 + 6s + 8) \\ \frac{Y(s)}{X(s)} &= \frac{s^3 + 4s^2 + 6s + 8}{s^3 + 3s^2 + s + 1} \end{aligned}$$

(b) If the initial condition is non zeros, we assume the initial conditions are $y(0), y(0)^{(1)}, y(0)^{(2)}$, then

$$\mathcal{L}(\frac{d^n y(t)}{dt^n}) = s^n Y(s) - \sum_{r=0}^{n-1} s^{n-r-1} y(0)^r$$

We have

$$\begin{aligned} & s^3Y(s) - (s^2y(0) + sy(0)^{(1)} + y(0)^{(2)}) + 3s^2Y(s) - 3(sy(0) + y(0)^{(1)}) + sY(s) - y(0) + Y(s) \\ & = s^3X(s) + 4s^2X(s) + 6xX(s) + 8X(s) \end{aligned}$$

Thus

$$\begin{aligned} Y(s) = & \underbrace{\frac{s^3 + 4s^2 + 6s + 8}{s^3 + 3s^2 + s + 1} X(s)}_{ZSR} \\ & + \underbrace{\frac{(s^2 + 3s + 1)y(0) + (s + 3)y(0)^{(1)} + y(0)^{(2)}}{s^3 + 3s^2 + s + 1}}_{ZIR} \end{aligned}$$

Problem 4

Let $x(t)$ be the sampled signal specified as

$$x(t) = \sum_{n=0}^{\infty} e^{-nT} \delta(t - nT)$$

where $T > 0$

(a) Determine $X(s)$, including its region of convergence.

(b) Sketeh the pole-zero plot for $X(s)$

(c) Use geometric interpretation of the pole-zero to argue that $X(j\Omega)$ is periodic.

Solution

(a) Note that

$$\delta(t-nT) \stackrel{c}{\longleftrightarrow} e^{-snT} \quad , \quad All\ s.$$

Therefore,

$$X(s) = \sum_{n=0}^{\infty} e^{-nT} e^{-snT} = \frac{1}{1 - e^{-T(1+s)}} \quad .$$

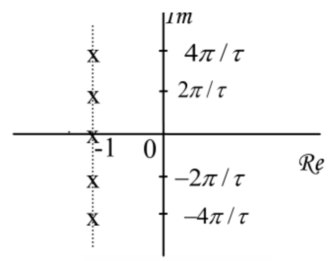


Figure S9.44

In order to determine the ROC, let us first find the poles of X(s) . Clearly , the poles occur when
This implies that the poles satisfy the following equation :

$$e^{-T(1+s_k)} = e^{jk2\pi}, k = 0, \pm 1, \pm 2, \cdots.$$

Taking the logarithm of both side of the above equation and simplifying , we get

$$s_k = -1 + \frac{jk2\pi}{T}, \qquad k = 0, \pm 1, \pm 2, \cdots.$$

Therefore , the poles all lie on a vertical line (parallel to the jw-axis) passing though s = -1 . Since the signal is right-sided , the ROC is Re{s}>-1.

(b)The pole-zero poles is as the figure.

(c)The magnitude of the Fourier transform $X(j\Omega)$ is given by the product of the reciprocal of the lengths of the vectors from the poles to the point jw .The phase of $X(j\Omega)$ is given by the negative of the sum of the angles of these vectors . Clearly from the pole-zero plot above it is clear that both the magnitude and phase have to vary periodically with a period of $\frac{2\pi}{T}$.

(d)

$$\begin{aligned} Y(s) &= \sum_{n=0}^{\infty} \cos(nT) \exp(-snT) \\ &= \frac{e^{T(j-s)} - e^{T(-j-s)}}{1 - e^{T(j-s)} - e^{T(-j-s)} + e^{-2Ts}} \ \text{ROC} : \mathcal{R}e[s] > 0. \end{aligned}$$

No zeros, poles: $s = \pm j + j\frac{2k\pi}{T}$.