Homework 7 Solutions

<u>Pr. 5.5</u> (a) The Laplace transforms are

$$x_1(t) = e^{-2t}u(t) \quad \Leftrightarrow \quad X_1(s) = \frac{1}{s+2} \quad \sigma > -2$$
$$x_2(t) = r(t) \quad \Leftrightarrow \quad X_1(s) = \frac{1}{s^2} \quad \sigma > 0$$
$$x_3(t) = te^{-2t}u(t) \quad \Leftrightarrow \quad X_1(s) = \frac{1}{(s+2)^2} \quad \sigma > -2$$

(b) The Laplace transforms of $x_1(t)$ and of $x_3(t)$ have regions of convergence containing the $j\Omega$-axis, and so we can find their Fourier transforms from their Laplace transforms by letting $s = j\Omega$
(c) The Fourier transforms of $x_1(t)$ and $x_3(t)$ are

$$X_1(\Omega) = \frac{1}{2+j\Omega}$$
$$X_3(\Omega) = \frac{1}{(2+j\Omega)^2}$$

Pr. 5.6 (a) In this case we are using the duality of the Fourier transforms so that the Fourier transform of the sinc is a pulse of magnitude A and cut-off frequency Ω_0 which we will need to determine.

The inverse Fourier transform is

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A[u(\Omega+\Omega_0) - u(\Omega-\Omega_0)]e^{j\Omega t}d\Omega \\ &= \frac{A}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{j\Omega t}d\Omega \\ &= \frac{A}{\pi t}\sin\Omega_0 t \end{aligned}$$

so that $A=\pi$ and $\Omega_0 = 1$, i.e.,

$$\frac{\sin(t)}{t} \quad \Leftrightarrow \quad \pi[u(\Omega+1) - u(\Omega-1)]$$

(b) The Fourier transform of $x_1(t) = u(t+0.5) - u(t-0.5)$ is

$$X_1(\Omega) = \left[\frac{1}{s}[e^{0.5s} - e^{-0.5s}]\right]_{s=j\Omega} = \frac{\sin(0.5\Omega)}{0.5\Omega}$$

Using the duality property we have:

$$\begin{aligned} x_1(t) = u(t+0.5) - u(t-0.5) \quad &\Leftrightarrow \quad X_1(\Omega) = \frac{\sin(\Omega/2)}{\Omega/2} \\ X_1(t) = \frac{\sin(t/2)}{t/2} \quad &\Leftrightarrow \quad 2\pi[u(\Omega+0.5) - u(\Omega-0.5)] \end{aligned}$$

using the fact that $x_1(t)$ is even. Then using the scaling property

$$\begin{aligned} X_1(2t) = \frac{\sin(t)}{t} \quad &\Leftrightarrow \quad \frac{2\pi}{2}[u((\Omega/2)+0.5) - u((\Omega/2)-0.5)] \\ &\Leftrightarrow \quad \pi[u(\Omega+1) - u(\Omega-1)] \end{aligned}$$

<u>**Pr. 5.7**</u>(a) The signal $x(t)$ is even while $y(t)$ is odd.
(b) The Fourier transform of $x(t)$ is

$$\begin{aligned} X(\Omega) &= \int_{-\infty}^{\infty} e^{-|t|}e^{-j\Omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-|t|}\cos(\Omega t)dt - j\int_{-\infty}^{\infty} e^{-|t|}\sin(\Omega t)dt \\ &= 2\int_{0}^{\infty} e^{-|t|}\cos(\Omega t)dt \end{aligned}$$

this is because the imaginary part is the integral of an odd function which is zero. Since $\cos(.)$ is an even function

$$X(-\Omega) = X(\Omega)$$

(c) The Fourier transform $X(\Omega)$ is

$$\begin{aligned} X(\Omega) &= 2\int_{0}^{\infty} e^{-t}\frac{e^{j\Omega t} + e^{-j\Omega t}}{2}dt \\ &= \int_{0}^{\infty} e^{-(1-j\Omega)t}dt + \int_{0}^{\infty} e^{-(1+j\Omega)t}dt \\ &= \frac{1}{1-j\Omega} + \frac{1}{1+j\Omega} = \frac{2}{1+\Omega^2} \end{aligned}$$

which is real-valued.
(d) For $y(t)$, odd function, its Fourier transform is

$$\begin{aligned} Y(\Omega) &= \int_{-\infty}^{\infty} y(t)e^{-j\Omega t}dt \\ &= -j\int_{-\infty}^{\infty} y(t)\sin(\Omega t)dt \end{aligned}$$

because $y(t)\cos(\Omega t)$ is an odd function and its integral is zero. The $Y(\Omega)$ is odd since

$$\begin{aligned} Y(-\Omega) &= -j\int_{-\infty}^{\infty} y(t)\sin(-\Omega t)dt \\ &= -Y(\Omega) \end{aligned}$$

since the sine is odd.
(e) Let's use the Laplace transform to find the Fourier transform of $y(t)$:

$$Y(s) = \frac{1}{s+1} - \frac{1}{-s+1}$$

with a region of convergence $-1 < \sigma < 1$, which contains the $j\Omega$-axis. So

$$Y(\Omega) = Y(s)\,|_{s=j\Omega} = \frac{1}{j\Omega+1} - \frac{1}{-j\Omega+1} = \frac{-2j\Omega}{1+\Omega^2}$$

which as expected is purely imaginary.
<u>Check:</u> Let $z(t) = x(t) + y(t) = 2e^{-t}u(t)$ which has a Fourier transform

$$Z(\Omega) = \frac{2}{1+j\Omega} = \frac{2(1-j\Omega)}{1+\Omega^2} = X(\Omega) + Y(\Omega)$$

(f) If a signal is represented as $x(t) = x_e(t) + x_o(t)$ then

$$X(\Omega) = X_e(\Omega) + X_o(\Omega)$$

where the first is a cosine transform and the second a sine transform.

Pr. 5.14 (a) (b) The Fourier series coefficients of $\delta_{T_s}(t)$ are

$$\Delta_k = \frac{1}{T_s}\mathcal{L}[\delta(t)]|_{s=jk\Omega_s} = \frac{1}{T_s}$$

so that

$$\delta_{T_s}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\Omega_s t} \qquad \Omega_s = \frac{2\pi}{T_s}$$

The FT of $\delta_{T_s}(t)$ is then

$$\begin{aligned}\Delta(\Omega) = \mathcal{F}[\delta_{T_s}(t)] &= \frac{1}{T_s}\sum_k \mathcal{F}[1e^{jk\Omega_s t}] \\ &= \frac{2\pi}{T_s}\sum_k \delta(\Omega - k\Omega_s)\end{aligned}$$

(c) Both $\delta_{T_s}(t)$ and $\Delta_{T_s}(\Omega)$ are periodic, the first of period T_s and the second of period $2\pi/T_s$.

Pr. 5.17 (a) The raised cosine is an even smooth signal with a value of 2 at the origin.
(b) The FT of the pulse $p(t) = u(t+1) - u(t-1)$ is

$$\begin{aligned} P(\Omega) &= \frac{1}{s}[e^{s} - e^{-s}] \mid_{s=j\Omega} \\ &= 2\frac{\sin(\Omega)}{\Omega} \end{aligned}$$

(c) The FT of

$$x(t) = (1 + \cos(2\pi t))p(t) = p(t) + p(t)\cos(2\pi t)$$

is

$$X(\Omega) = P(\Omega) + \frac{1}{2}[P(\Omega - 2\pi) + P(\Omega + 2\pi)]$$

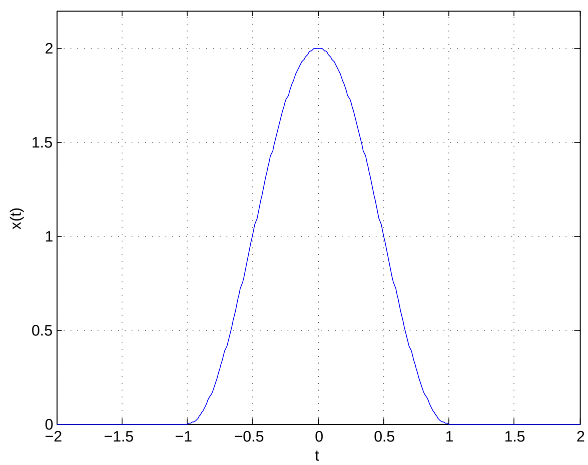

Figure 5.7: Raised cosine $x(t)$