Homework 2 Solution

EE 140: Introduction to Communication Systems

Problem 1 (12 points)

For a lowpass signal with a bandwidth of 6000 Hz
(a) What is the minimum sampling frequency for perfect reconstruction of the signal?
(b) What is the minimum required sampling frequency if a guard band of 2000 Hz is required?
(c) What is the minimum required sampling frequency and the value of K for perfect reconstruction if the reconstruction filter has the following frequency response

$$H(f) = \begin{cases} K & |f| < 7000 \\ K - K\frac{|f|-7000}{3000} & 7000 < |f| < 10000 \\ 0 & \text{otherwise} \end{cases}$$

Solution 1

(a) For no aliasing to occur we must sample at the Nyquist rate

$$f_s = 2 \cdot 6000 \text{ samples/sec} = 12000 \text{ samples/sec}$$

(b) With a guard band of 2000 Hz

$$f_s - 2W = 2000 \Longrightarrow f_s = 14000$$

(c) The reconstruction filter should not pick-up frequencies of the images of the spectrum $X(f)$. The nearest image spectrum is centered at f_s and occupies the frequency band $[f_s - W, f_s + W]$. Thus the highest frequency of the reconstruction filter ($= 10000$) should satisfy

$$10000 \leq f_s - W \Longrightarrow f_s \geq 16000$$

For the value $f_s = 16000$, K should be such that

$$K \cdot f_s = 1 \Longrightarrow K = (16000)^{-1}$$

Problem 2 (13 points)

The lowpass signal $x(t)$ with a bandwidth of W is sampled at the Nyquist rate and the signal

$$x_1(t) = \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t - nT_s)$$

is generated. Find the Fourier transform of $x_1(t)$.

Solution 2

$$\begin{aligned}
x_1(t) = & \sum_{n=-\infty}^{\infty} (-1)^n x(nT_s)\delta(t - nT_s) = x(t) \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - nT_s) \\
& = x(t)[\sum_{l=-\infty}^{\infty} \delta(t - 2lT_s) - \sum_{l=-\infty}^{\infty} \delta(t - T_s - 2lT_s)] \\
X_1(f) = & X(f) * [\tfrac{1}{2T_s} \sum_{l=-\infty}^{\infty} \delta(f - \tfrac{l}{2T_s}) - \tfrac{1}{2T_s} \sum_{l=-\infty}^{\infty} \delta(f - \tfrac{l}{2T_s}) e^{-j2\pi f T_s}] \\
& = \tfrac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \tfrac{l}{2T_s}) - \tfrac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \tfrac{l}{2T_s}) e^{-j2\pi \frac{l}{2T_s} T_s} \\
& = \tfrac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \tfrac{l}{2T_s}) - \tfrac{1}{2T_s} \sum_{l=-\infty}^{\infty} X(f - \tfrac{l}{2T_s})(-1)^l \\
& = \tfrac{1}{T_s} \sum_{l=-\infty}^{\infty} X(f - \tfrac{1}{2T_s} - \tfrac{l}{T_s})
\end{aligned}$$

Problem 3 (10 points)

Two binary random variables X and Y are distributed according to the joint distribution $p(X = Y = 0) = p(X = 0, Y = 1) = p(X = Y = 1) = \frac{1}{3}$. Compute $H(X), H(Y), H(X|Y), H(X,Y)$ and $I(X;Y)$.

Solution 3

The marginal probabilities are given by

$$P(X=0)=\sum_k p(X=0,Y=k)=p(X=0,Y=0)+p(X=0,Y=1)=\frac{2}{3}$$

$$p(X=1)=\sum_k p(X=1,Y=k)=p(X=1,Y=1)=\frac{1}{3}$$

$$p(Y=0)=\sum_k (X=k,Y=0)=p(X=0,Y=0)=\frac{1}{3}$$

$$p(Y=1)=\sum_k (X=k,Y=1)=p(X=0,Y=1)+p(X=1,Y=1)=\frac{2}{3}$$

Hence,

$$H(X)=-\sum p_i\log_2 p_i=-(\frac{1}{3}\log_2\frac{1}{3}+\frac{2}{3}\log_2\frac{2}{3})=.9183$$

$$H(Y)=-\sum p_i\log_2 p_i=-(\frac{1}{3}\log_2\frac{1}{3}+\frac{2}{3}\log_2\frac{2}{3})=.9183$$

$$H(X,Y)=-\sum p_i\log_2 p_i=-(\frac{1}{3}\log_2\frac{1}{3}+\frac{1}{3}\log_2\frac{1}{3}+\frac{1}{3}\log_2\frac{1}{3})=1.5850$$

$$H(X|Y)=H(X,Y)-H(Y)=1.5850-0.9183=0.6667$$

$$I(X;Y)=H(X)-H(X|Y)=0.9183-0.6667=0.2516$$

Problem 4 (10 points)

A signal can be modeled as a lowpass stationary process $X(t)$ whose PDF at any time t_0

is given in Figure 1.

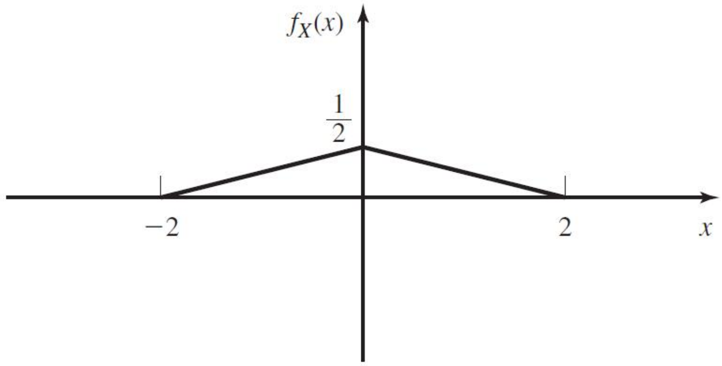

Figure 1

The bandwidth of this process is 5 KHz, and it is desired to transmit it using a PCM system.
(a) If sampling is done at the Nyquist rate and a uniform quantizer with 32 levels is employed, what is the resulting SQNR? What is the resulting bit rate?
(b) If the available bandwidth of the channel is 40 KHz, what is the highest achievable SQNR?

Solution 4

(a)

$$\begin{aligned}
E[X^2(t)] &= \int_{-2}^{0} x^2(\frac{x+2}{4})dx + \int_{0}^{2} x^2(\frac{-x+2}{4})dx \\
&= \frac{1}{4}(\frac{1}{4}x^4 + \frac{2}{3}x^3)|_{-2}^{0} + \frac{1}{4}(-\frac{1}{4}x^4 + \frac{2}{3}x^3)|_{0}^{2} \\
&= \frac{2}{3}
\end{aligned}$$

Hence,

$$\text{SQNR} = \frac{3 \times 4^{\nu} \times \frac{2}{3}}{x_{\max}^2} = \frac{3 \times 4^5 \times \frac{2}{3}}{2^2} = 512 = 27.093(db)$$

(b)

If the available bandwidth of the channel is 40 KHz, then the maximum rate of transmission is $\nu = 40/5 = 8$. In this case the highest achievable SQNR is

$$\text{SQNR} = \frac{3 \times 4^8 \times \frac{2}{3}}{2^2} = 32768 = 45.154(db)$$

Problem 5 (15 points)

Show that a pulse having the raised cosine spectrum given by Equation (1) satisfies the Nyquist criterion given by Equation (2) for any value of the roll-off factor α.

$$X_{rc}(f) = \begin{cases} T \ , & 0 \leq |f| \leq (1-\alpha)/2T \\ \frac{T}{2}[1 + \cos(\frac{\pi T}{\alpha}(|f| - \frac{1-\alpha}{2T}))] \ , & \frac{1-\alpha}{2T} \leq |f| \leq \frac{1+\alpha}{2T} \\ 0 \ , & |f| \geq \frac{1+\alpha}{2T} \end{cases} \tag{1}$$

$$\sum_{m=-\infty}^{\infty} X(f + \frac{m}{T}) = T \tag{2}$$

Solution 5

The pulse $x(t)$ having the raised cosine spectrum is

$$x(t) = sinc(t/T) \frac{\cos(\pi \alpha t/T)}{1 - 4\alpha^2 t^2/T^2}$$

The function $sinc(t/T)$ is 1 when $t = 0$ and 0 when $t = nT$. On the other hand

$$g(t) = \frac{\cos(\pi \alpha t/T)}{1 - 4\alpha^2 t^2/T^2} = \begin{cases} 1 & t = 0 \\ \text{bounded} & t \neq 0 \end{cases}$$

The function $g(t)$ needs to be checked only for those values of t such that $4\alpha^2 t^2/T^2 = 1$ or $\alpha t = \frac{T}{2}$.

However,

$$\lim_{\alpha t \to \frac{T}{2}} \frac{\cos(\pi \alpha t/T)}{1 - 4\alpha^2 t^2/T^2} = \lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x}$$

and by using L'Hospital's rule

$$\lim_{x \to 1} \frac{\cos(\frac{\pi}{2}x)}{1-x} = \lim_{x \to 1} \frac{\pi}{2}\sin(\frac{\pi}{2}x) = \frac{\pi}{2} < \infty$$

Hence,

$$x(nT) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

meaning that the pulse $x(t)$ satisfies the Nyquist criterion.

Problem 6 (15 points)

Binary PAM is used to transmit information over an unequalized linear filter channel. When $a = 1$ is transmitted the noise-free output of the demodulator is

$$x_m = \begin{cases} 0.3 \quad , & m = 1 \\ 0.9 \quad , & m = 0 \\ 0.3 \quad , & m = -1 \\ 0 \quad\quad , & \text{otherwise} \end{cases} \tag{3}$$

(a) Design a three-tap zero-forcing linear equalizer so that the output is

$$q_m = \begin{cases} 1 \ , & m = 0 \\ 0 \ , & m = \pm 1 \end{cases} \tag{4}$$

(b) Determine q_m for $m = \pm 2, \pm 3$, by convolving the impulse response of the equalizer with the channel response

Solution 6

(a)

The equivalent discrete-time impulse response of the channel is

$$h(t) = \sum_{n=-1}^{1} h_n \delta(t - nT) = 0.3\delta(t + T) + 0.9\delta(t) + 0.3\delta(t - T)$$

If by c_n we denote the coefficients of the FIR equalizer, then the equalized signal is

$$q_m = \sum_{n=-1}^{1} c_n h_{m-n}$$

which in matrix notation is written as

$$\begin{pmatrix} 0.9 & 0.3 & 0 \\ 0.3 & 0.9 & 0.3 \\ 0 & 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The coefficients of the zero-force equalizer can be found by solving the previous matrix equation.

Thus,

$$\begin{pmatrix} c_{-1} \\ c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} -0.4762 \\ 1.4286 \\ -0.4762 \end{pmatrix}$$

(b)

The values of q_m for $m = \pm 2, \pm 3$ are given by

$$q_2 = \sum_{n=-1}^{1} c_n h_{2-n} = c_1 h_1 = -0.1429$$

$$q_{-2} = \sum_{n=-1}^{1} c_n h_{-2-n} = c_{-1} h_{-1} = -0.1429$$

$$q_3 = \sum_{n=-1}^{1} c_n h_{3-n} = 0$$

$$q_{-3} = \sum_{n=-1}^{1} c_n h_{-3-n} = 0$$

Problem 7 (10 points)

Find the capacity of an additive white Gaussian noise channel with a bandwidth of 1 MHz, power of 10W, and noise power-spectral density of $\frac{N_0}{2} = 10^{-9}$ W/Hz.

Solution 7

The SNR is

$$SNR = \frac{2P}{N_0 2W} = \frac{P}{2W} = \frac{10}{10^{-9} \times 10^6} = 10^4$$

Thus the capacity of the channel is

$$C = W \log_2(1 + \frac{P}{N_0 W}) = 10^6 \log_2(1 + 10000) \approx 13.2879 \times 10^6 \text{bits/sec}$$

Problem 8 (15 points)

For the channel shown in Figure 2, find the channel capacity and the input distribution that achieves capacity.

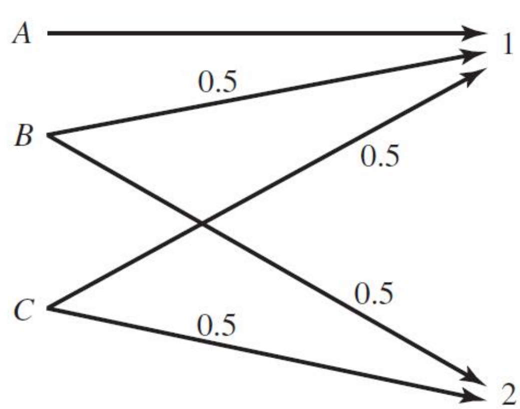

Figure 2

Solution 8

The capacity of the channel of the channel is given by

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)}[H(Y) - H(Y|X)]$$

Let the probability of the inputs C, B and A be p, q and $1-p-q$ respectively. From the symmetry of the nodes B, C, we expect that the optimum distribution $p(x)$ will satisfy $p(B) = p(C) = p$. The entropy $H(Y|X)$ is given by

$$\begin{aligned} H(Y|X) &= \sum p(x)H(Y|X=x) = (1-2p)H(Y|X=A) + 2pH(Y|X=B) \\ &= 0 + 2ph(0.5) = 2p \end{aligned}$$

The probability mass function of the output is

$$p(Y=1) = \sum p(x)p(Y=1|X=x) = (1-2p) + p = 1-p$$

$$p(Y=2) = \sum p(x)p(Y=2|X=x) = 0.5p + 0.5p = p$$

Therefore,

$$C = \max_p[H(Y) - H(Y|X)] = \max_p(h(p) - 2p)$$

To find the optimum value of p that maximizes $I(X;Y)$, we set the derivative of C with respect to p equal to zero. Thus,

$$\begin{aligned} \frac{\partial C}{\partial p} = 0 &= -\log_2(p) - p\frac{1}{p\ln(2)} + \log_2(1-p) - (1-p)\frac{-1}{(1-p)\ln(2)} - 2 \\ &= \log_2(1-p) - \log_2(p) - 2 \end{aligned}$$

and therefore

$$\log_2\frac{1-p}{p} = 2 \Rightarrow \frac{1-p}{p} = 4 \Rightarrow p = \frac{1}{5}$$

The capacity of the channel is

$$C = h(\frac{1}{5}) - \frac{2}{5} = 0.7219 - 0.4 = 0.3219 \text{bits/transmission}$$