Multi-Period Portfolio Optimization for Index Tracking in Finance

Xiuyuan Huang, Zepeng Zhang, and Ziping Zhao
School of Information Science and Technology, ShanghaiTech University, Shanghai 201210, China
Email: {huangxy1, zhangzp1, zhaoziping}@shanghaitech.edu.cn

***Abstract*—Index tracking, a classical passive investment strategy in finance, attempts to reproduce the performance of a specific market index by holding only a subset of the constituent assets in the index. To realize this, various portfolio optimization methods have been developed. In the literature, all the existing works focus on single-period optimization (SPO) for index tracking portfolio design. However, in the financial markets, such SPO methods may lead to frequent portfolio rebalances, resulting in high transaction costs. In this paper, a novel multi-period optimization (MPO) approach to index tracking portfolio design is proposed, which is able to account for transaction costs and holding costs. The MPO for index tracking is formulated as a nonconvex optimization problem and solved successively by dealing with a second-order cone programming subproblem based on the successive convex optimization procedure. Numerical simulations showcase that the proposed MPO method is able to achieve comparative tracking performance with lower costs compared to the classical SPO method.**

***Index Terms*—Multi-period portfolio, index tracking, asset selection, successive convex approximation.**

I. Introduction

In the financial industry, investment strategies can be categorized as active management and passive management [1]. Active fund managers attempt to find those “diamond in the rough” assets based on their expertise and judgment to construct a portfolio whose values are going to outperform the markets. While the passive fund managers aim at achieving similar performance with the market benchmarks (i.e., the financial indices). Passive management has aroused much interest in recent years since historical data has shown that most actively managed funds failed to outperform the markets [2], while the passively managed funds can make decent profits when the market rises (which is true in the long run historically) and can mitigate the idiosyncratic risks associated with various companies. A straightforward way of passive investment is to hold the same proportion of asset shares as the index, leading to the so-called full replication strategy. However, such a strategy may involve too many illiquid assets which will translate into high risk. Besides, when the index is revised, portfolio rebalancing may cause excessive transaction costs [3]. Therefore, passive fund managers attempt to reproduce the performance of an index by holding only a subset of the assets in an index, leading to the index tracking portfolio (ITP) design problem [4].

The target of the ITP design problem is to minimize the tracking error while achieving a “sparse” portfolio, which can be realized either by introducing extra binary variables to limit the portfolio size directly [5], [6], [7] or by penalizing the cardinality into the portfolio design objectives [8]. For example, in [5], the tracking error is modeled from a regression-based point of view and extra binary variables are introduced to limit the portfolio size. In [4], the empirical tracking error measured by the mean squared error is adopted and the sparsity is promoted with a nonconvex approximation of the cardinality function (i.e., the ℓ_0-“norm” [9]). However, all the existing works focus on the single-period optimization (SPO) for ITP design, that is, the portfolios to be invested in multiple periods are optimized independently, making such myopic strategies sub-optimal, say, causing high transaction costs [10]. Similar problems exist in many other portfolio management problems, to mitigate which, the multi-period optimization (MPO) method has been brought up [11], [12].

MPO strategies for portfolio design have received lots of popularity as it not only can account for transaction costs over multiple periods but also can handle the conflicting return estimates on different time scales [12]. In [13], the MPO strategy is adopted to fulfill the objective of minimizing a cumulative risk measure over the investment horizon, in the meantime satisfying portfolio diversity constraints at each period and achieving a desired amount of terminal wealth. The MPO method has also been applied in other portfolio design problems such as the mean-variance portfolio design [14] and the robust semi-variance downside risk portfolio design [15]. Besides, the MPO scheme recently was adopted into the risk parity portfolio design [16].

In this paper, we attempt to take advantage of the MPO method to address the aforementioned issues in the financial index tracking portfolio design problems. To the best of our knowledge, this is the first work that introduces the MPO scheme into the ITP design. To realize asset selection, i.e., selecting only a subset of assets in an index, a nonconvex constraint that promotes sparsity is also introduced into the problem. The successive convex approximation (SCA) algorithm [17] is used to solve the resulting nonconvex optimization problem, where the inner convex approximation subproblem is solved as a second-order cone program (SOCP). In the end, the benefits of introducing MPO into ITP design are demonstrated through numerical simulations on the market data.

II. Index Tracking Portfolio

Given an index composed of n assets, we denote at time τ the return of the index as $r_{\text{ind},\tau}$ and returns of the index constituents plus the cash return (denoted as the $(n+1)$th asset) as $\mathbf{r}_\tau = [r_{1,\tau}, \ldots, r_{n,\tau}, r_{n+1,\tau}]^\mathsf{T}$. Given the portfolio weights

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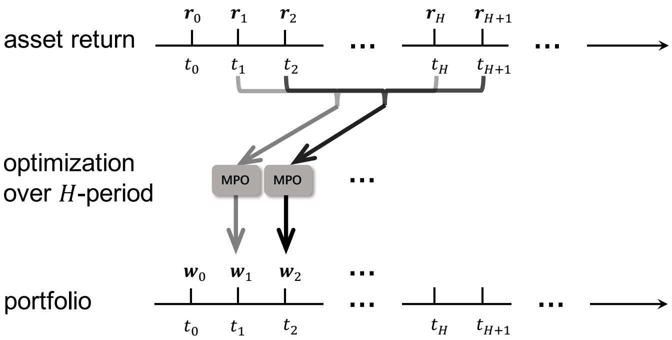

Figure 1. The H-period multi-period optimization scheme.

$\mathbf{w}_\tau = [w_{1,\tau}, \ldots, w_{n,\tau}, w_{n+1,\tau}]^\mathsf{T}$ satisfying $\mathbf{1}^T\mathbf{w}_\tau = 1$, the return of the ITP at time τ is defined as $\mathbf{r}_\tau^\mathsf{T}\mathbf{w}_\tau$. In this paper, short-selling is not allowed for the index constituent assets (i.e., $w_{i,\tau} \geq 0$ for $i = 1, \ldots, n$).

The return tracking error of the designed ITP w.r.t. the index at time τ can be chosen as one of the following commonly used forms:

$$g_{\text{TE}}(\mathbf{w}_\tau) = \begin{cases} |r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T}\mathbf{w}_\tau| & = g_{\text{ATE}}(\mathbf{w}_\tau) \\ |r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T}\mathbf{w}_\tau|^2 & = g_{\text{STE}}(\mathbf{w}_\tau) \\ (r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T}\mathbf{w}_\tau)_+ & = g_{\text{DTE}}(\mathbf{w}_\tau) \\ (r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T}\mathbf{w}_\tau)_+^2 & = g_{\text{SDTE}}(\mathbf{w}_\tau), \end{cases}$$

where $g_{\text{ATE}}(\mathbf{w}_\tau)$, $g_{\text{STE}}(\mathbf{w}_\tau)$, $g_{\text{DTE}}(\mathbf{w}_\tau)$, and $g_{\text{SDTE}}(\mathbf{w}_\tau)$ represent the absolute tracking error (ATE), the squared tracking error (STE), the downside tracking error (DTE), and the squared downside tracking error (SDTE), respectively, and $(x)_+ \triangleq \max\{x, 0\}$.

In the markets, inevitable transaction costs, or trading costs, are involved along with any trading actions. Hence, frequent rebalancing of the portfolio is not desired in practice as it will lead to high trading costs. To take such costs into account, a trading cost function for the constituent asset i between time $\tau - 1$ and time τ is modeled as follows [12]:

$$\phi_{\text{trade}}(d_{i,\tau}) = a_i|d_{i,\tau}| + b_i\sigma_{i,\tau}\frac{|d_{i,\tau}|^{3/2}}{(V_{i,\tau}/B_\tau)^{1/2}} + c_i d_{i,\tau}, \quad (1)$$

where a_i, b_i, c_i are predefined parameters, $\mathbf{d}_\tau = \mathbf{w}_\tau - \mathbf{w}_{\tau-1}$ denote the trading actions at time τ, $\sigma_{i,\tau}$ is the price volatility at time τ, $V_{i,\tau}$ represents the total market volume of asset i at time τ, and B_τ represents the net portfolio value at time τ. Holding costs are associated with storing unsold inventory. In this paper, we only consider the holding costs generated from loans [12], which is defined as

$$\phi_{\text{hold}}(w_{n+1,\tau}) = s(-w_{n+1,\tau})_+, \quad (2)$$

where s is the given shorting cost rate.

To obtain sparse portfolios (i.e., $w_{i,\tau} = 0$ for some i), we consider adding the following nonconvex constraint [18], [19] into the design problem to limit the number of active assets in the portfolio:

$$\sum_{i=1}^{n} \frac{w_{i,\tau}}{\eta + w_{i,\tau}} \leq \kappa,$$

where $\eta > 0$ is a given small constant and $\kappa > 0$ controls the maximal number of assets to be selected.

Assuming that no external cash can be put into or taken out of the portfolio during the investment, the portfolio rebalance from period $\tau - 1$ to period τ should satisfy the self-financing constraint expressed as

$$\mathbf{1}^\mathsf{T}\mathbf{d}_\tau + \sum_{i=1}^{n+1} \phi_{\text{trade}}(d_{i,\tau}) + \phi_{\text{hold}}(w_{n+1,\tau}) = 0.$$

In practice, trading costs and holding costs are extremely small and neglectable compared to the total portfolio value. It is reasonable to simplify this constraint as $\mathbf{1}^\mathsf{T}\mathbf{d}_\tau = 0$ [12].

A. Single-Period Optimization for ITP Design

By considering both the tracking performance and the exceeding returns, we get the following objective function

$$f(\mathbf{w}_\tau) = g_{\text{TE}}(\mathbf{w}_\tau) - \gamma_{\text{return}}\mathbf{r}_\tau^\mathsf{T}\mathbf{w}_\tau,$$

where $g_{\text{TE}}(\mathbf{w}_\tau)$ generally denotes any tracking error function whose specification can be chosen according to the practice condition, and $\gamma_{\text{return}} \geq 0$ is a predefined parameter trading off the tracking errors and the exceeding returns. Finally, the SPO model for ITP design (SPO-ITP) is given as follows:

$$\begin{aligned} \underset{\mathbf{w}_\tau, \mathbf{d}_\tau}{\text{minimize}} \quad & f(\mathbf{w}_\tau) \\ & + \gamma_{\text{trade}}\sum_{i=1}^{n+1}\phi_{\text{trade}}(d_{i,\tau}) + \gamma_{\text{hold}}\phi_{\text{hold}}(w_{n+1,\tau}) \\ \text{subject to} \quad & w_{i,\tau} \geq 0, \ \text{ for } i = 1, \ldots n \\ & \sum_{i=1}^{n}\frac{w_{i,\tau}}{\eta + w_{i,\tau}} \leq \kappa \\ & \mathbf{d}_\tau = \mathbf{w}_\tau - \mathbf{w}_{\tau-1} \\ & \mathbf{1}^\mathsf{T}\mathbf{d}_\tau = 0, \end{aligned} \qquad \text{(SPO-ITP)}$$

where $\gamma_{\text{trade}} \geq 0$ and $\gamma_{\text{hold}} \geq 0$ are predefined constants.

Remark 1. In Problem (SPO-ITP), the prediction of values of the returns at time τ, i.e., $r_{\text{ind},\tau}$ and $\mathbf{r}_\tau$, is an important task in ITP design [20]. However, it is not the focus of this paper and we will assume them to be perfectly known in this paper.

B. Multi-Period Optimization for ITP Design

Under the SPO paradigm, portfolios over multiple periods will be optimized independently, which ignores the potential influence induced by current portfolios on the future ones. Therefore, there is a chance that the portfolio constructed based on the current condition may perform unfavorably in the future and we have to suffer from the inevitable loss or alleviate the loss by frequently rebalancing the portfolios, which brings in high transaction costs. To overcome such limitations in SPO methods, an MPO model for the ITP design (MPO-ITP) is proposed in this paper. Under the MPO scheme, the current portfolio is determined by solving an optimization problem that minimizes the cumulative tracking error over H periods with $H > 1$, where the transaction costs and holding costs are also included. The process of a sequence of portfolios constructed under the MPO scheme is shown in Figure 1,

$$\begin{aligned}
\underset{\{\mathbf{w}_\tau\},\ \{\mathbf{d}_\tau\}}{\text{minimize}} \quad & \sum_{\tau=t+1}^{t+H} f(\mathbf{w}_\tau) + \gamma_{\text{trade}} \sum_{i=1}^{n+1} \left(a_i |d_{i,\tau}| + \frac{b_i \sigma_{i,\tau}}{(V_{i,\tau}/B_\tau)^{1/2}} |d_{i,\tau}|^{3/2} + c_i d_{i,\tau} \right) + \gamma_{\text{hold}} s(-w_{n+1,\tau})_+ \\
\text{subject to} \quad & w_{i,\tau} \geq 0, \ \text{ for } i = 1, \dots n, \ \forall \tau \\
& \sum_{i=1}^{n} \frac{w_{i,\tau}^{(0)}}{\eta + w_{i,\tau}^{(0)}} + \frac{\epsilon}{(\eta + w_{i,\tau}^{(0)})^2} (w_{i,\tau} - w_{i,\tau}^{(0)}) \leq \kappa, \ \forall \tau \\
& \mathbf{d}_\tau = \mathbf{w}_\tau - \mathbf{w}_{\tau-1}, \ \forall \tau \\
& \mathbf{1}^\mathsf{T} \mathbf{d}_\tau = 0, \ \forall \tau
\end{aligned} \tag{3}$$

where $\mathbf{w}_\tau$ is determined by solving an optimization problem based on the information of the next H periods.

Specifically, the H-period MPO-ITP problem is modeled as

$$\begin{aligned}
\underset{\{\mathbf{w}_\tau\}, \{\mathbf{d}_\tau\}}{\text{minimize}} \quad & \sum_{\tau=t+1}^{t+H} f(\mathbf{w}_\tau) \\
& + \gamma_{\text{trade}} \sum_{i=1}^{n+1} \phi_{\text{trade}}(d_{i,\tau}) + \gamma_{\text{hold}} \phi_{\text{hold}}(w_{n+1,\tau}) \\
\text{subject to} \quad & w_{i,\tau} \geq 0, \ \text{ for } i = 1, \dots n, \ \forall \tau \\
& \sum_{i=1}^{n} \frac{w_{i,\tau}}{\eta + w_{\tau,i}} \leq \kappa, \ \forall \tau \\
& \mathbf{d}_\tau = \mathbf{w}_\tau - \mathbf{w}_{\tau-1}, \ \forall \tau \\
& \mathbf{1}^\mathsf{T} \mathbf{d}_\tau = 0, \ \forall \tau.
\end{aligned} \tag{MPO-ITP}$$

Problem (MPO-ITP) reduces to Problem (SPO-ITP) if $H = 1$. The MPO-ITP problem is nonconvex owing to the nonconvex sparsity promoting constraint. In this paper, an efficient algorithm based on the SCA algorithm [17] will be developed for problem solving.

III. Solving the MPO-ITP Problem via SCA

In this section, to solve the nonconvex MPO-ITP problem, we develop an SCA-based algorithm, which solves a sequence of convex surrogate problems iteratively. To realize efficient resolution, the convex inner approximation problem is further transformed into an SOCP to be solved by standard SOCP solvers like MOSEK [21].

A. Convexification of the Sparsity Promoting Constraint

We introduce the following useful lemma.

Lemma 2. *[22], [23] Function $\sum_{i=1}^{n} \frac{w_{i,\tau}}{\eta + w_{i,\tau}}$ is concave when $w_{i,\tau} \geq 0$, $i = 1, \dots, n$, in which case a linear upperbound can be constructed at $w_{i,\tau}^{(0)}$, $i = 1, \dots, n$, as follows:*

$$\sum_{i=1}^{n} \frac{w_{i,\tau}}{\eta + w_{i,\tau}} \leq \sum_{i=1}^{n} \frac{w_{i,\tau}^{(0)}}{\eta + w_{i,\tau}^{(0)}} + \frac{\eta}{(\eta + w_{i,\tau}^{(0)})^2} (w_{i,\tau} - w_{i,\tau}^{(0)}),$$

where the equality is attained when $w_{i,\tau} = w_{i,\tau}^{(0)}$ for all i.

By substituting the trading cost function $\phi_{\text{trade}}(d_{i,\tau})$ and the holding cost function $\phi_{\text{hold}}(w_{n+1,\tau})$ into (MPO-ITP) and leveraging on Lemma 2, Problem (MPO-ITP) can be solved by iteratively solving Problem (3), which is convex and can be readily solved via the general-purpose solvers like CVX [24].

B. An SOCP Reformulation of Problem (3)

In this section, we will transform Problem (3) into a standard SOCP to evoke a more efficient resolution procedure. To achieve this, we first introduce the following two results.

Lemma 3. *The minimization of $g_{\text{TE}}(\mathbf{w}_\tau)$ over $\mathbf{w}_\tau$ can be equivalently formulated into the following standard SOCP:*

$$\begin{aligned}
\underset{\mathbf{w}_\tau,\, v_\tau,\, h_\tau}{\textit{minimize}} \quad & h_\tau \\
\textit{subject to} \quad & h_\tau \in \mathcal{H}_\tau,
\end{aligned}$$

where

$$\mathcal{H}_\tau \triangleq \begin{cases}
\left\{ |r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T} \mathbf{w}_\tau| \leq h_\tau \right\}, & \textit{for } g_{\text{ATE}}(\mathbf{w}_\tau) \\
\left\{ r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T} \mathbf{w}_\tau \leq v_\tau, \ v_\tau^2 \leq h_\tau \right\}, & \textit{for } g_{\text{STE}}(\mathbf{w}_\tau) \\
\left\{ r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T} \mathbf{w}_\tau \leq h_\tau, \ 0 \leq h_\tau \right\}, & \textit{for } g_{\text{DTE}}(\mathbf{w}_\tau) \\
\left\{ r_{\text{ind},\tau} - \mathbf{r}_\tau^\mathsf{T} \mathbf{w}_\tau \leq v_\tau, \ 0 \leq v_\tau, \ v_\tau^2 \leq h_\tau \right\}, & \\
\qquad \textit{for } g_{\text{SDTE}}(\mathbf{w}_\tau). &
\end{cases}$$

Lemma 4. *The minimization of $|d_{i,\tau}|^{3/2}$ over $d_{i,\tau}$ can be equivalently formulated into the following standard SOCP:*

$$\begin{aligned}
\underset{d_{i,\tau},\, q_{i,\tau},\, p_{i,\tau},\, y_{i,\tau}}{\textit{minimize}} \quad & y_{i,\tau} \\
\textit{subject to} \quad & |d_{i,\tau}| \leq q_{i,\tau} \\
& \left\| \begin{matrix} 2q_{i,\tau} \\ p_{i,\tau} - y_{i,\tau} \end{matrix} \right\|_2 \leq p_{i,\tau} + y_{i,\tau} \\
& \left\| \begin{matrix} 2p_{i,\tau} \\ q_{i,\tau} - 1 \end{matrix} \right\|_2 \leq q_{i,\tau} + 1 \\
& 0 \leq p_{i,\tau} \\
& 0 \leq y_{i,\tau}.
\end{aligned} \tag{4}$$

Proof: The proof is given in Appendix A. ∎

Based on Lemma 3 and Lemma 4, Problem (3) can be reformulated into a standard SOCP as given in Problem (SOCP) to be solved in the kth iteration, by iteratively solving which we can obtain a solution for Problem (MPO-ITP). The overall algorithm is outlined in Algorithm 1.

IV. Numerical Simulations

To validate the performance of our proposed model, the following experiment is designed. We collect market data of S&P 500 from Yahoo! Finance for one year from April 1st, 2020 to March 31st, 2021. We calculate the historical returns of the S&P 500 index and the $n = 503$ assets as the predictions for $r_{\text{ind},\tau}$ and $\mathbf{r}_\tau$, and choose $f_{\text{SDTE}}(\mathbf{w}_\tau)$ as the tracking error measure. The data from April 1st, 2020 to April 30th, 2020 is chosen as the training set which is used to tune

$$\begin{aligned} \underset{\mathcal{X}}{\text{minimize}} \quad & \sum_{\tau=t+1}^{t+H} h_\tau - \gamma_{\text{return}} \mathbf{r}_\tau^\mathsf{T} \mathbf{w}_\tau + \gamma_{\text{trade}} \sum_{i=1}^{n+1} [a_i q_{i,\tau} + \frac{b_i \sigma_{i,\tau}}{(V_{i,\tau}/B_\tau)^{1/2}} y_{i,\tau} + c_i d_{i,\tau}] + \gamma_{\text{hold}} s l_\tau \triangleq F(\mathcal{X}) \\ \text{subject to} \quad & \mathcal{X} \in \mathbb{X}^{(k)} \end{aligned} \tag{SOCP}$$

with $\mathcal{X} \triangleq \{\mathbf{w}_\tau, \mathbf{d}_\tau, v_\tau, h_\tau, \{q_{i,\tau}\}, \{p_{i,\tau}\}, \{y_{i,\tau}\}, l_\tau\}$ where l_τ is an auxiliary variable and the following defined constraint

$$\left.\begin{aligned} & w_{i,\tau} \geq 0, \ \text{ for } i = 1, \dots n, \ \forall \tau \\ & \sum_{i=1}^{n} \frac{w_{i,\tau}^{(k)}}{\eta + w_{i,\tau}^{(k)}} + \frac{\eta}{(\eta + w_{i,\tau}^{(k)})^2} (w_{i,\tau} - w_{i,\tau}^{(k)}) \leq \kappa, \ \forall \tau \\ & \mathbf{d}_\tau = \mathbf{w}_\tau - \mathbf{w}_{\tau-1}, \ \mathbf{1}^\mathsf{T} \mathbf{d}_\tau = 0, \ \forall \tau \\ & h_\tau \in \mathcal{H}_\tau, \ \forall \tau \\ & |d_{i,\tau}| \leq q_{i,\tau}, \ \forall i, \forall \tau \\ & \left\| \begin{matrix} 2q_{i,\tau} \\ p_{i,\tau} - y_{i,\tau} \end{matrix} \right\|_2 \leq p_{i,\tau} + y_{i,\tau}, \ \left\| \begin{matrix} 2p_{i,\tau} \\ q_{i,\tau} - 1 \end{matrix} \right\|_2 \leq q_{i,\tau} + 1, \ \forall i, \forall \tau \\ & 0 \leq p_{i,\tau}, \ 0 \leq y_{i,\tau}, \ \forall i, \forall \tau \\ & -w_{n+1,\tau} \leq l_\tau, \ 0 \leq l_\tau, \ \forall i, \forall \tau \end{aligned}\right\} \triangleq \mathbb{X}^{(k)}$$

Algorithm 1 The SCA Algorithm for MPO-ITP Design.

Require: H, $r_{\text{ind},\tau}$, $\mathbf{r}_\tau$, $\{\sigma_{i,\tau}\}$, $\{V_{i,\tau}\}$, $(\tau = t+1, \dots, t+H)$, $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, B_τ, s, γ_{return}, γ_{trade}, γ_{hold}, η, κ, and ϵ

Set $k = 0$, and $\mathbf{w}^{(0)}$.
repeat
  $\mathcal{X}^{(k+1)} = \arg \min_{\mathcal{X} \in \mathbb{X}^{(k)}} F(\mathcal{X})$ (i.e., solving an SOCP)
  $k \leftarrow k + 1$
until convergence

parameters by cross validation, and the data from May 1st, 2020 to March 31st, 2021 is chosen as the test set. For the trading cost function, we approximate the daily volatility with $\sigma_{i,\tau} = |\log p_{i,\tau}^{\text{open}} - \log p_{i,\tau}^{\text{close}}|$, where $p_{i,\tau}^{\text{open}}$ and $p_{i,\tau}^{\text{close}}$ denote the open price and close price of asset i at time τ, and choose $a_i = 0.05\%$, $b_i = 1$, $c_i = 0$ for all i and τ. For the holding cost function, $s = 0.01\%$ is set as the shorting cost rate. For hyperparameters, we set $H = 5$, $\gamma_{\text{trade}} = 5$, $\gamma_{\text{hold}} = 5$, $\gamma_{\text{return}} = 0.001$, $\eta = 10^{-4}$, and $\kappa = 30$. We assign the initial portfolio value as $B_0 = 1,000,000$ (in dollars) and the initial portfolio weights as $\mathbf{w}_0 = [0, \dots, 0, 1]^\mathsf{T}$. To solve the inner SOCP problems, we use MOSEK [21] and in each iteration, the algorithm will stopped when $\|\mathbf{w}^{(\text{new})} - \mathbf{w}^{(\text{old})}\|_1 \leq \epsilon$, where we set $\epsilon = 10^{-6}$.

The experiment is designed to compare the performance of the proposed MPO method and the classical SPO method. For each trading day, we optimize the trading strategies for the next H trading days, with only the next trading day executed, and we continuously optimize the portfolios for eleven months to test the cumulative performance of the strategies. The cumulative costs (including both trading costs and holding costs) of SPO and MPO are shown in Figure 2, which demonstrates that the MPO method can lead to lower costs compared to the SPO counterpart over the eleven-month period. Next, the index return and the cumulative returns of SPO and MPO are depicted in Figure 3. It shows that both SPO and MPO can outperform the index with MPO achieving slightly higher return performance than SPO.

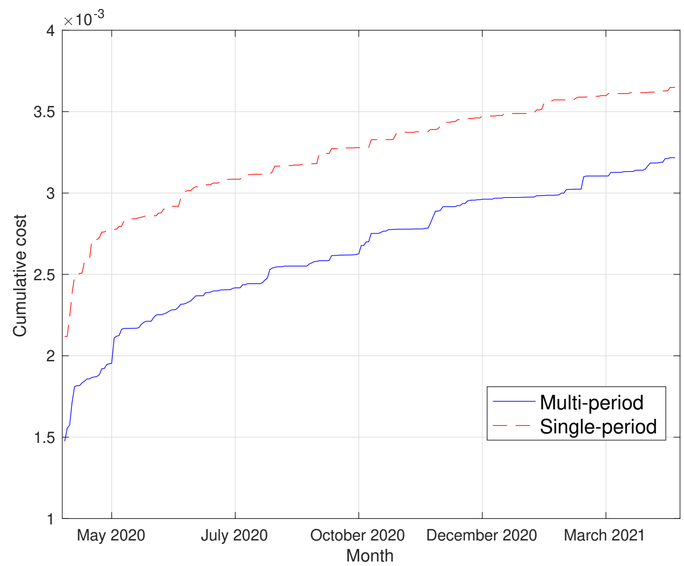

Figure 2. The cumulative cost comparison between MPO and SPO.

Table I
RUNNING TIME COMPARISON FOR MPO-ITP DESIGNS BY SOLVING PROBLEM (3) AND PROBLEM (SOCP) IN SECONDS (S)

(H, κ)	(1, 15)	(1, 30)	(5, 15)	(5, 30)	(10, 15)	(10, 30)
CVX	10.19s	7.52s	85.34s	55.3s	146.35s	99.19s
MOSEK	4.13s	4.01s	22.10s	19.22s	90.34s	49.96s

We further compare the running time of the SCA algorithm with the inner problems solved either by CVX or by the MOSEK solver. Comparison results with different parameters of H and κ are shown in Table I. As expected, solving the inner convex problems directly via MOSEK can make the algorithm more efficient and scalable than solving it via CVX.

V. Conclusions

In this paper, a multi-period optimization model for index tracking portfolio design has been proposed. The resulting

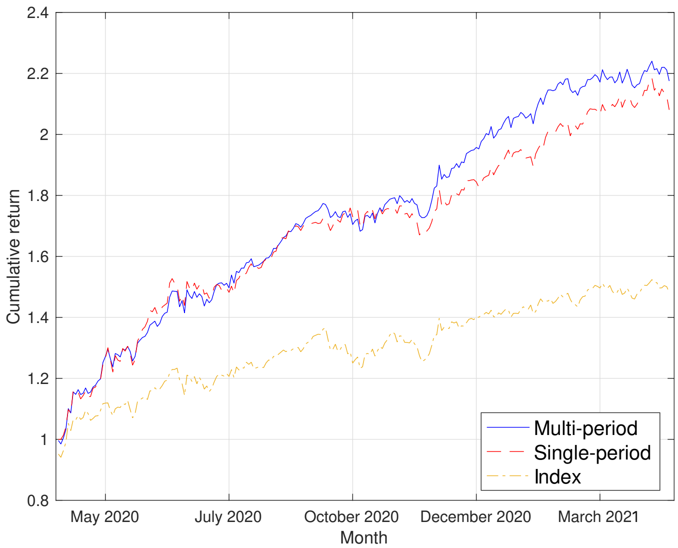

Figure 3. The cumulative return comparison between MPO, SPO, and S&P 500 index.

nonconvex portfolio design problem is tackled via successive convex approximation method, with which the problem can be addressed by successively solving an SOCP subproblem. Numerical simulations demonstrate that the proposed multi-period method achieves comparable tracking performance with lower costs compared to the single-period method, showing that the proposed method is more favorable for practical financial index tracking targets.

Appendix A
Proof of Lemma 4

Proof: We first have the following equivalent transforms:

$$\underset{d_{i,\tau}}{\text{minimize}} \quad |d_{i,\tau}|^{3/2} \quad \Leftrightarrow \quad \begin{array}{ll} \underset{d_{i,\tau},\, q_{i,\tau}}{\text{minimize}} & q_{i,\tau}^{3/2} \\ \text{subject to} & |d_{i,\tau}| \le q_{i,\tau} \end{array}$$

$$\Leftrightarrow \begin{array}{ll} \underset{d_{i,\tau},\, q_{i,\tau},\, y_{i,\tau}}{\text{minimize}} & y_{i,\tau} \\ \text{subject to} & |d_{i,\tau}| \le q_{i,\tau} \\ & q_{i,\tau}^{3/2} \le y_{i,\tau} \end{array} \Leftrightarrow \begin{array}{ll} \underset{d_{i,\tau},\, q_{i,\tau},\, y_{i,\tau}}{\text{minimize}} & y_{i,\tau} \\ \text{subject to} & |d_{i,\tau}| \le q_{i,\tau} \\ & q_{i,\tau}^2 \le y_{i,\tau} q_{i,\tau}^{1/2} \\ & 0 \le y_{i,\tau} \end{array}$$

$$\Longleftrightarrow \begin{array}{ll} \underset{d_{i,\tau},\, q_{i,\tau},\, p_{i,\tau},\, y_{i,\tau}}{\text{minimize}} & y_{i,\tau} \\ \text{subject to} & |d_{i,\tau}| \le q_{i,\tau} \\ & q_{i,\tau}^2 \le y_{i,\tau} p_{i,\tau} \\ & p_{i,\tau}^2 \le q_{i,\tau} \\ & 0 \le p_{i,\tau} \\ & 0 \le y_{i,\tau}. \end{array}$$

Based on the result that the rotated scalar Lorentz cone $\{w^2 \le uv,\ u \ge 0,\ v \ge 0)\}$ can be written as a standard SOCP constraint

$$\left\| \begin{array}{c} 2w \\ u - v \end{array} \right\|_2 \le u + v. \tag{5}$$

We obtain

$$q_{i,\tau}^2 \le y_{i,\tau} p_{i,\tau} \Leftrightarrow \left\| \begin{array}{c} 2q_{i,\tau} \\ p_{i,\tau} - y_{i,\tau} \end{array} \right\|_2 \le p_{i,\tau} + y_{i,\tau},$$

and

$$p_{i,\tau}^2 \le q_{i,\tau} \Leftrightarrow \left\| \begin{array}{c} 2p_{i,\tau} \\ q_{i,\tau} - 1 \end{array} \right\|_2 \le q_{i,\tau} + 1,$$

which completes the proof. ■

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