

Detection & Estimation

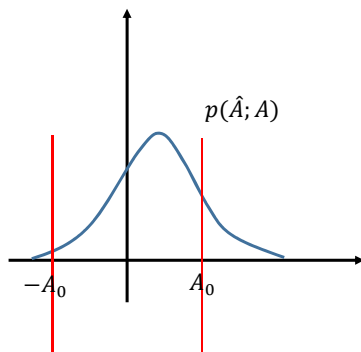
Lecture 4

Bayesian

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Prior Knowledge

- Physical constraint: $A \in [-A_0, A_0]$



$$\hat{A} = \frac{1}{N} \sum x[n]$$

$$\check{A} = \begin{cases} -A_0, & \hat{A} < -A_0 \\ \hat{A}, & \hat{A} \in [-A_0, A_0] \\ A_0, & \hat{A} > A_0 \end{cases}$$

$$mse(\hat{A}) > mse(\check{A})$$

Prior Knowledge

- Knowing that the unknown parameter must lie in a known interval, we may assign a uniform PDF and model the true value as a realization of a random variable
- Bayesian MSE

$$Bmse(\hat{A}) = E_{A,x} [(A - \hat{A})^2]$$

Bmse vs MSE

$$Bmse(\hat{A}) = \iint (A - \hat{A})^2 p(x, A) dx dA$$

$$mse(\hat{A}) = \int (\hat{A} - A)^2 p(x; A) dx$$

Minimum Bmse Estimator

$$Bmse(\hat{A}) = \int \left[\int (A - \hat{A})^2 p(A|x) dA \right] p(x) dx$$

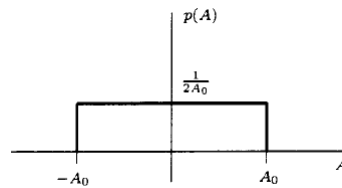


minimize

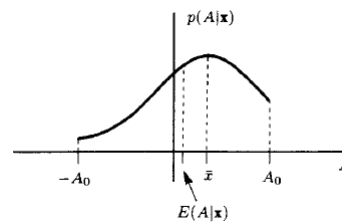
$$\hat{A} = E[A|x] = \int A p(A|x) dA$$

Posterior vs Prior PDF

$$p(A) = \int p(x, A) dx$$

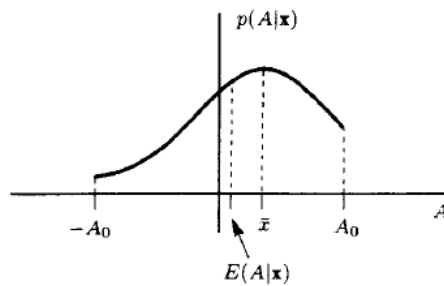


$$p(A|x) = \frac{p(x, A)}{p(x)}$$



Posterior PDF

$$p(A|\mathbf{x}) = \begin{cases} \frac{\frac{1}{2A_0(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A)^2\right]}{\int_{-A_0}^{A_0} \frac{1}{2A_0(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - A)^2\right] dA} & |A| \leq A_0 \\ 0 & |A| > A_0. \end{cases}$$



Gaussian Prior

$$p(A) = \begin{cases} \frac{1}{2A_0} & |A| \leq A_0 \\ 0 & |A| > A_0 \end{cases} \quad \text{uniform prior}$$

$$p(A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left[-\frac{1}{2\sigma_A^2}(A - \mu_A)^2\right] \quad \text{Gaussian prior}$$

Gaussian Prior

$$\begin{aligned}
 p(A|\mathbf{x}) &= \frac{\exp\left[-\frac{1}{2\sigma_{A|x}^2}(A - \mu_{A|x})^2\right] \exp\left[-\frac{1}{2}\left(\frac{\mu_A^2}{\sigma_A^2} - \frac{\mu_{A|x}^2}{\sigma_{A|x}^2}\right)\right]}{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_{A|x}^2}(A - \mu_{A|x})^2\right] \exp\left[-\frac{1}{2}\left(\frac{\mu_A^2}{\sigma_A^2} - \frac{\mu_{A|x}^2}{\sigma_{A|x}^2}\right)\right] dA} \\
 &= \frac{1}{\sqrt{2\pi\sigma_{A|x}^2}} \exp\left[-\frac{1}{2\sigma_{A|x}^2}(A - \mu_{A|x})^2\right]
 \end{aligned}$$

$$\begin{aligned}
 \hat{A} &= \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}} \bar{x} + \frac{\frac{\sigma^2}{N}}{\sigma_A^2 + \frac{\sigma^2}{N}} \mu_A \\
 &= \alpha \bar{x} + (1 - \alpha) \mu_A
 \end{aligned}$$

$$\alpha = \frac{\sigma_A^2}{\sigma_A^2 + \frac{\sigma^2}{N}}$$

Gaussian PDF Properties

Theorem 10.1 (Conditional PDF of Bivariate Gaussian) If x and y are distributed according to a bivariate Gaussian PDF with mean vector $[E(x) E(y)]^T$ and covariance matrix

$$\mathbf{C} = \begin{bmatrix} \text{var}(x) & \text{cov}(x, y) \\ \text{cov}(y, x) & \text{var}(y) \end{bmatrix}$$

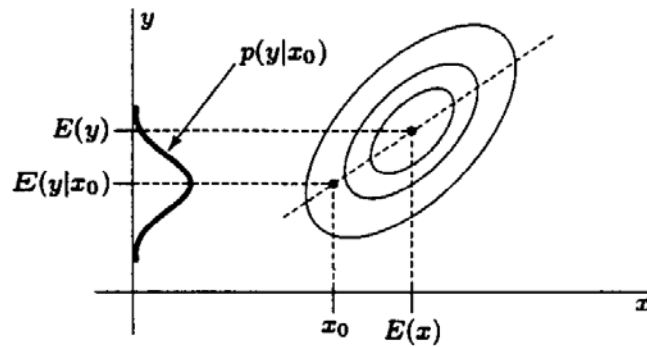
so that

$$p(x, y) = \frac{1}{2\pi \det^{\frac{1}{2}}(\mathbf{C})} \exp\left[-\frac{1}{2} \begin{bmatrix} x - E(x) \\ y - E(y) \end{bmatrix}^T \mathbf{C}^{-1} \begin{bmatrix} x - E(x) \\ y - E(y) \end{bmatrix}\right],$$

then the conditional PDF $p(y|x)$ is also Gaussian and

$$\begin{aligned}
 E(y|x) &= E(y) + \frac{\text{cov}(x, y)}{\text{var}(x)} (x - E(x)) \\
 \text{var}(y|x) &= \text{var}(y) - \frac{\text{cov}^2(x, y)}{\text{var}(x)}.
 \end{aligned}$$

Gaussian PDF Properties



HW

- Chapter 10
 - 10.1, 10.2, 10.3, 10.4, 10.5, 10.6, 10.10, 10.11