SPARSE UNMIXING BASED ON FEATURE PIXELS FOR HYPERSPECTRAL IMAGERY

Xinrui Cui and Z. Jane Wang

University of British Columbia
Department of Electrical and Computer Engineering
Vancouver, BC V6T 1Z4, Canada

ABSTRACT

Here, a novel sparse unmixing approach for hyperspectral scene is proposed. Through the analysis of the difference of mixing degree in pixels, feature pixels are defined as a set of pixels where the linear combination of the whole endmembers is able to represent any pixel in the scene. Namely, endmembers for the whole scene can be obtained through only feature pixels. Through the geological property of simplex formed by hyperspectral data, it can be known that feature pixels are vertices of the simplex. Based on N-FINDER algorithm, the feature pixels can be identified. Then, the feature pixels are decomposed in linear sparse unmixing algorithm to obtain corresponding endmembers. Finally, least square method is applied to estimate the abundance of endmembers in the whole hyperspectral scene. Experimental results demonstrate the efficacy and accuracy of the proposed algorithm.

***Index Terms*—** Sparse unmixing, feature pixel, vertex

1. INTRODUCTION

Hyperspectral remote sensing is focused on the measurement, analysis and interpretation of spectra acquired by an airborne or satellite hyperspectral sensor [1]. Generally, the spectrum of interest locates in wavelength from 0.4 μm to 2.5 μm, covering from visible to infrared spectral bands [2]. With hundreds of spectral channels, remotely sensed hyperspectral imaging has a very high spectral resolution. However, due to the insufficient spatial resolution of imaging spectrometers, almost each pixel in the hyperspectral scene contains more than one pure substance. To obtain accurate estimation of substances, unmixing is proposed and it aims at decomposing the measured spectrum of each mixed pixel into a combination of pure spectral signatures (endmembers) [3] and a set of corresponding fractions (abundances) [4].

Typically, unmixing models can be classified as nonlinear and linear. Nonlinear unmixing models assume that part of the source radiation is multiply scattered before being collected by the sensor. Conversely, linear unmixing models assume minimal secondary reflections and multiple scattering in the data collection procedure and thus hold several advantages such as ease of implementation and flexibility in diverse applications [5]. Specifically, the linear analysis expresses the measured spectrum of each mixed pixel as a linear combination of endmembers weighted by abundances that indicate the proportion of each endmember in the pixel.

Tremendous effort has been put to the linear unmixing models since past years. In general, linear unmixing algorithm can be classified as semisupervised algorithm and unsupervised one depending on whether using the spectral library or not. For the unsupervised approaches, several methods based on statistics [6, 7], nonnegative matrix factorization (NMF) [8, 9] and geometry [10, 11, 12, 13] have been developed. As these approaches extract endmembers merely from the hyperspectral data, they either could obtain virtual endmembers with no physical meaning [14] or assume the presence of at least one pure pixel per endmember in the data, which is usually difficult to guarantee [15]. Taking the spectral library as a priori knowledge, a semi-supervised approach, sparse unmixing [16, 17], is proposed, which bypasses the limits of unsupervised approach. It aims at using only a few spectral signatures in a given spectral library to model each mixed pixel in the hyperspectral scene. As the number of actual endmembers in a hyperspectral scene is usually much smaller than the number of spectral signatures in the library, this approach often leads to a sparse solution. Based on the increasingly mature linear sparse representation techniques [18], several effective linear sparse unmixing algorithms have been proposed such as the sparse unmixing via variable splitting and augmented Lagrangian (SUnSAL) and a constrained version of the same algorithm (CSUnSAL) [19], which are based on the alternating direction method of multipliers (ADMM) [20] in a way similar to previous articles [21, 22]. However, one great challenge faced by the sparse unmixing algorithms is the high mutual coherence of spectral library, i.e. the largest cosine between any two spectral signatures in the library.

To propose a more effective sparse unmixing algorithm based on spectral library, here we analyze the difference of degree of mixing in pixels, and use the difference to implement more efficient identification of endmembers. Specifically, in the real hyperspectral scene, part of mixed pixels have relatively low degree of mixing, which are more similar to pure endmembers and thus more representative. If a minimal subset of such representative mixed pixels can be found that any pixels in the scene can be expressed as a linear combination of them, then pixels in the subset are called feature pixels. Meanwhile, the feature pixels can be expressed as a linear combination of pure endmembers in spectral library. Therefore, any pixels in the hyperspectral scene also can be expressed as a linear combination of endmembers obtained from the feature pixels. Namely, we can identify all endmembers from hyperspectral library through only feature pixels instead of all pixels in a scene. Through least square method, the abundance value of each identified endmember can be easily calculated.

The rest of the article is structured as follows. In Section 2, we present the proposed sparse unmixing algorithm. In Section 3, experiments are implemented to evaluate our algorithm. Finally, the conclusion is shown in Section 4.

2. THE PROPOSED METHOD

2.1. The Meaning of Feature Pixels

Suppose $\mathbf{A} \in \mathbb{R}^{L\times m}$ is the given spectral library, where L is the number of spectral bands and m is the number of spectral signatures in the library; $\boldsymbol{y} \in \mathbb{R}^L$ is the spectrum vector of a mixed pixel; $\boldsymbol{x} \in \mathbb{R}^m$ is the abundance vector with regard to the library $\mathbf{A}$. Then the sparse unmixing model is

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{n} \in \mathbb{R}^L$ is the vector of error term.

The model has the following two constraints:

$$\boldsymbol{x} \geq 0 \tag{2}$$

$$\sum_{i=1}^{m} x_i = 1 \tag{3}$$

which are called abundance nonnegativity constraint and sum-to-one constraint, respectively [23].

In feature space, hyperspectral data could form a simplex. Generally, a L-simplex is a L-dimensional polytope which is the convex hull of its $L+1$ vertices; any point inside can be expressed as a linear combination of vertices. Based on the property, feature pixels can be regarded as vertices of simplex.

Consider $\mathbf{V}$ as the set of vertices, and it can be expressed as

$$\mathbf{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_{L+1}] \tag{4}$$

where $\boldsymbol{v}_i \in \mathbb{R}^L$ represents the i-th vertex.

Then any point inside the simplex is

$$\begin{aligned} \boldsymbol{y} &= \sum_{i=1}^{L+1} \theta_i \boldsymbol{v}_i \\ s.t. \quad & \theta_i \geqslant 0, \sum_{i=1}^{L+1} \theta_i = 1 \end{aligned} \tag{5}$$

where θ_i is the abundance value of $\boldsymbol{v}_i$.

Suppose $\mathbf{E} = [\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_P]$, a subset of $\mathbf{A}$, is the set of endmembers corresponding to vertices, which means

$$\begin{aligned} \boldsymbol{v}_i &= \sum_{j=1}^{P} x_i^j \boldsymbol{e}_j \\ s.t. \quad & x_i^j \geqslant 0, \sum_{j=1}^{P} x_i^j = 1 \end{aligned} \tag{6}$$

where x_i^j is the abundance value of j-th endmember corresponding to i-th vertex.

According to Eq. (5) and (6), we can get

$$\begin{aligned} \boldsymbol{y} &= \sum_{i=1}^{L+1} \theta_i (\sum_{j=1}^{P} x_i^j \boldsymbol{e}_j) \\ &= \sum_{i=1}^{L+1} \theta_i (x_i^1 \boldsymbol{e}_1 + x_i^2 \boldsymbol{e}_2 + \cdots + x_i^P \boldsymbol{e}_P) \\ &= \sum_{i=1}^{L+1} (\theta_i x_i^1 \boldsymbol{e}_1 + \theta_i x_i^2 \boldsymbol{e}_2 + \cdots + \theta_i x_i^P \boldsymbol{e}_P) \\ &= \sum_{i=1}^{L+1} \theta_i x_i^1 \boldsymbol{e}_1 + \sum_{i=1}^{L+1} \theta_i x_i^2 \boldsymbol{e}_2 + \cdots + \sum_{i=1}^{L+1} \theta_i x_i^P \boldsymbol{e}_P \end{aligned} \tag{7}$$

Since $\theta_i \geqslant 0$, $x_i^j \geqslant 0$, then $\sum_{i=1}^{L+1} \theta_i x_i^j \geqslant 0$.

Meanwhile, $\sum_{i=1}^{L+1} \theta_i = 1$ and $\sum_{j=1}^{P} x_i^j = 1$, thus the sum of coefficients in Eq. (7) is

$$\begin{aligned} & \sum_{i=1}^{L+1} \theta_i x_i^1 + \sum_{i=1}^{L+1} \theta_i x_i^2 + \cdots + \sum_{i=1}^{L+1} \theta_i x_i^P \\ &= \sum_{i=1}^{L+1} \theta_i (x_i^1 + x_i^2 + \cdots + x_i^P) \\ &= \sum_{i=1}^{L+1} \theta_i \\ &= 1 \end{aligned} \tag{8}$$

The linear Eq. (7) meets both abundance nonnegativity constraint and sum-to-one constraint. Therefore, in hyperspectral scene, any mixed pixel can be linearly expressed by endmembers obtained merely from feature pixels.

2.2. The Identification of Feature Pixels

To identify feature pixels in a hyperspectral scene, we need to extract vertices of the simplex. Winter [24] first proposed an endmember extraction approach through finding the maximum volume data closing simplex, which results in a widely applied algorithm in hyperspectral image analysis — N-FINDR. Although various versions of N-FINDR have been proposed, such as sequential N-FINDR [25] and random N-FINDR [26], they share the similar principle, namely, finding a set of vectors from the original data which could comprise a simplex with maximum volume.

The optimization formulation of the principle could be written as follows

$$\begin{aligned} &\underset{\boldsymbol{v}_1,\cdots,\boldsymbol{v}_\lambda\in\mathbb{R}^{\lambda-1}}{max} \operatorname{vol}(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_\lambda) \\ &s.t. \quad \boldsymbol{v}_i \in \mathcal{S} = \operatorname{conv}\{\tilde{\boldsymbol{y}}_1,\cdots,\tilde{\boldsymbol{y}}_N\}, i=1,\cdots,N. \end{aligned} \tag{9}$$

where N represents the number of pixels in the hyperspectral scene, $\tilde{\boldsymbol{y}}_i$ is the observed i-th column vector of $\tilde{\mathbf{Y}}$ (usually dimension-reduced), $\mathcal{S} = \operatorname{conv}\{\}$ indicates that the vectors could comprise a convex hull, and $\operatorname{vol}(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_\lambda)$ calculates the volume of simplex that is captured by the vectors. The volume is mathematically written as:

$$\operatorname{vol}(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_\lambda) = \frac{1}{(\lambda-1)!}|\det(\begin{bmatrix}\hat{\mathbf{V}}\\ \mathbf{1}^T\end{bmatrix})| \tag{10}$$

where $\hat{\mathbf{V}} = [\boldsymbol{v}_1,\cdots,\boldsymbol{v}_\lambda]$ and $\mathbf{1}^T$ is an all-one value column vector.

With a reasonable assumption that the origin of coordinate is also a vertex, an equivalent form for calculating volume is popular and shown as follows [27]:

$$\operatorname{vol}(\boldsymbol{v}_1,\cdots,\boldsymbol{v}_\lambda) = \frac{1}{(\lambda-1)!}\sqrt{\det(\mathbf{B}^T\mathbf{B})} \tag{11}$$

where $\mathbf{B} = (\boldsymbol{v}_\lambda - \boldsymbol{v}_1, \boldsymbol{v}_\lambda - \boldsymbol{v}_2, \cdots, \boldsymbol{v}_\lambda - \boldsymbol{v}_{\lambda-1})$.

Then we will decompose feature pixels in linear unmixing model.

2.3. Sparse Unmixing Using Feature Pixels

In this stage, the endmembers of feature pixels are obtained by SUnSAL which is an effective linear sparse unmixing algorithm. Then, mixed pixels in the scene are decomposed into the obtained endmembers and their corresponding abundances. Least square method is used to get the abundance of the obtained endmembers for the whole hyperspectral scene.

For the i-th vertex, the $\boldsymbol{v}_i$ in the set of vertices $\mathbf{V}$, the optimization problem of sparse unmixing can be written as follows

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_0 \quad \text{subject to} \quad \|\boldsymbol{v}_i - \mathbf{A}\boldsymbol{x}\|_2 \le \delta, \boldsymbol{x} \ge 0 \tag{12}$$

where $\|\boldsymbol{x}\|_0$ denotes the number of nonzero components in $\boldsymbol{x}$, $\delta \ge 0$ is the error tolerance due to the noise and modeling errors.

With an appropriate Lagrange multiplier, the problem above is equivalent to

$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{v}_i - \mathbf{A}\boldsymbol{x}\|_2^2 + \lambda_1\|\boldsymbol{x}\|_0 \quad \text{subject to} \quad \boldsymbol{x} \ge 0 \tag{13}$$

where $\lambda_1 \ge 0$ denotes regularization parameter. However, the problem in Eq. (13) is NP-hard, which means it can not be solved in polynomial time. Typically, the l_1 norm could be a better alternative of l_0 norm for sparse unmixing. Therefore the problem in Eq. (13) can be converted into

$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{v}_i - \mathbf{A}\boldsymbol{x}\|_2^2 + \lambda_1\|\boldsymbol{x}\|_1 \quad \text{subject to} \quad \boldsymbol{x} \ge 0 \tag{14}$$

Based on ADMM algorithm, we can solve the above optimization problem and get the endmembers of feature pixels [19]. Then least square method is used to get the abundances of the obtained endmembers for all mixed pixels.

3. EXPERIMENTS

In this section, experiments are conducted to test the effectiveness of the proposed approach in comparison with SUnSAL.

3.1. Performance Discriminators

The root mean square error (RMSE) is used to evaluate the abundance estimations. For the i-th endmember, RMSE is defined as

$$\text{RMSE}_i \equiv \sqrt{\frac{1}{N}\sum_{j=1}^{N}(\alpha_i^j - \hat{\alpha}_i^j)^2} \tag{15}$$

where α_i^j represents the true abundance value of the i-th endmember in the j-th pixel and $\hat{\alpha}_i^j$ is the estimated abundance.

RMSE_i measures the quality of the reconstruction of the fractional abundances of the i-th endmember in all pixels. The mean value of all the endmembers' RMSEs will be calculated. In general, smaller RMSE means more accurate estimation.

3.2. Simulated Data Sets

Here, we evaluate the performances of the proposed algorithms in different type of noises, different signal-to-noise ratios (SNR $\equiv 10\log_{10}\frac{\|\mathbf{AX}\|_2^2}{\|\mathbf{n}\|_2^2}$) of noise and different endmember numbers. Specifically, the synthetic data are corrupted by Gaussian white noise and correlated noise[1] with different levels of SNR: 20, 30 and 40 dB.

The spectral library we use in our experiment is the first part of the United States Geological Survey (USGS) [28]. Specifically, the spectral library $\mathbf{A} \in R^{224\times498}$ contains 498 spectral signatures with 224 spectral bands distributed uniformly in the interval 0.4–2.5 μm. Nine spectral signatures

[1]The Gaussian white noise is generated using the *awgn* function in MATLAB. The correlated noise is generated using the *correlatedGaussianNoise* function that is available online: http://www.mathworks.com/matlabcentral/fileexchange/21156-correlated-gaussian-noise/content/correlatedGaussianNoise.m. The correlation matrix is set as default.

Table 1. RMSEs obtained by different algorithms on the simulated hyperspectral data (white noise)

	SNR (dB)	SUnSAL	OURS
SD1 ($k_1 = 3$)	20	0.0676	**0.0443**
	30	0.0225	**0.0104**
	40	0.0074	**0.0057**
SD2 ($k_2 = 6$)	20	0.0679	**0.0507**
	30	0.0231	**0.0133**
	40	0.0079	**0.0063**
SD3 ($k_3 = 9$)	20	0.0695	**0.0582**
	30	0.0277	**0.0197**
	40	0.0264	**0.0170**

Table 2. RMSEs obtained by different algorithms on the simulated hyperspectral data (correlated noise)

	SNR (dB)	SUnSAL	OURS
SD1 ($k_1 = 3$)	20	0.0697	**0.0473**
	30	0.0266	**0.0125**
	40	0.0088	**0.0061**
SD2 ($k_2 = 6$)	20	0.0723	**0.0517**
	30	0.0284	**0.0156**
	40	0.0095	**0.0065**
SD3 ($k_3 = 9$)	20	0.0730	**0.0563**
	30	0.0397	**0.0199**
	40	0.0316	**0.0168**

are chosen from $\mathbf{A}$ to generate the synthetic hyperspectral image: Rhodochrosite HS67, Axinite HS342.3B, Chrysocolla HS297.3B, Niter GDS43 (K-Saltpeter), Anthophyllite HS286.3B, Neodymium Oxide GDS34, Monazite HS255.3B, Samarium Oxide GDS36 and Pigeonite HS199.3B. Finally, following a Dirichlet distribution [13], the simulated data sets, each of which contains 900 pixels, are generated using different endmember numbers: $k_1 = 3$, $k_2 = 6$, $k_3 = 9$. To make sure that no pure pixel exists in the hyperspectral data, we force all the abundances to be no larger than 0.7.

3.3. Experiment Analysis

Tab. 1 shows the RMSEs obtained by different algorithms on the simulated data corrupted by white noise. The performances of all the algorithms degrade as the noise gets stronger and the endmember number increases. It can be seen that in most cases, the proposed algorithm outperforms SUnSAL. This phenomenon indicates that using feature pixels can improve the performance of sparse unmixing algorithm.

Tab. 2 shows the results obtained on the simulated data corrupted by correlated noise. Since the noise in the real hyperspectral images is usually correlated, this case is closer to the practical ones. From Tab. 2 we can see that in most cases, the proposed algorithm behaves better than the SUnSAL.

4. CONCLUSIONS

In the article, a novel sparse unmixing algorithm for hyperspectral imagery is proposed. Feature pixels are introduced that the spectral vector of each mixed pixel is a linear combination of endmembers of feature pixels. Therefore, endmembers for the whole scene can be obtained through only feature pixels. Specifically, the feature pixels are decomposed in linear sparse unmixing algorithm to obtain endmembers from hyperspectral library. Then, least square method is applied to estimate the abundance of obtained endmembers in the whole hyperspectral scene. Experimental results indicate that the proposed algorithm possesses improvement in accuracy compared with SUnSAL.

5. ACKNOWLEDGEMENT

The authors would like to thank J. Bioucas-Dias and M. Figueiredo for sharing their codes for SUnSAL.

6. REFERENCES

[1] A. Plaza, J. Benediktsson, J. Boardman, J. Brazile, L. Bruzzone, G. Camps-Valls, J. Chanussot, M. Fauvel, P. Gamba, A. Gualtieri, M. Marconcini, J. Tilton, and G. Trianni. "Recent advances in techniques for hyperspectral image processing," *Remote Sens. Env.*, vol. 113, pp. 110–122, 2009.

[2] R. Green, M. Eastwood, C. Sarture, T. Chrien, M. Aronsson, B. Chippendale, J. Faust, B. Pavri, C. Chovit, M. Solis, M. R. Olah, and O. Williams. "Imaging spectroscopy and the airborne visible/infrared imaging spectrometer (AVIRIS)," *Remote Sens. Env.*, vol. 65, no. 3, pp. 227–248, 1998.

[3] A. Plaza, P. Martinez, R. Perez, and J. Plaza, "A quantitative and comparative analysis of endmember extraction algorithms from hyperspectral data," *IEEE Trans. Geosci. Remote Sens.*, vol. 42, no. 3, pp. 650–663, 2004.

[4] D. Heinz and C.-I. Chang, "Fully constrained least squares linear mixture analysis for material quantification in hyperspectral imagery," *IEEE Trans. Geosci. Remote Sens.*, vol. 39, no. 3, pp. 529–545, 2001.

[5] Y. Hu, H. Lee, and F. Scarpace, "Optimal linear spectral unmixing," *IEEE Trans. Geosci. Remote Sens.*, vol. 37, no. 1, pp. 639–644, 1999.

[6] M. Berman, H. Kiiveri, R. Lagerstrom, A. Ernst, R. Dunne, and J. Huntington, "ICE: A statistical approach to identifying endmembers in hyperspectral images," *IEEE Trans. Geosci. Remote Sens.*, vol. 42, no. 10, pp. 2085–2095, 2004.

[7] F. Schmidt, A. Schmidt, E. Treandguier, M. Guiheneuf, S. Moussaoui, and N. Dobigeon, "Implementation strategies for hyperspectral unmixing using Bayesian source separation," *IEEE Trans. Geosci. Remote Sens.*, vol. 48, no. 11, pp. 4003–4013, 2010.

[8] D. Seung and L. Lee, "Algorithms for non-negative matrix factorization," *NIPS*, vol. 13, pp. 556–562, 2001.

[9] V. P. Pauca, J. Piper, and R. J. Plemmons, "Nonnegative matrix factorization for spectral data analysis," *Linear Algebra Appl.*, vol. 416, no. 1, pp. 29–47, 2006.

[10] M. Winter, "N-FINDR: An algorithm for fast autonomous spectral endmember determination in hyperspectral data," *Proc. SPIE*, vol. 3753, pp. 266–277, 1999.

[11] M. Zortea and A. Plaza, "A quantitative and comparative analysis of different implementations of N-FINDR: A fast endmember extraction algorithm," *IEEE Geosci. Remote Sens. Lett.*, vol. 6, no. 4, pp. 787–791, 2009.

[12] A. Zare and P. Gader, "PCE: Piecewise convex endmember detection," *IEEE Trans. Geosci. Remote Sens.*, vol. 48, no. 6, pp. 2620–2632, 2010.

[13] J. M. Nascimento and J. Bioucas-Dias, "Vertex component analysis: A fast algorithm to unmix hyperspectral data," *IEEE Trans. Geosci. Remote Sens.*, vol. 43, no. 4, pp. 898–910, 2005.

[14] X. Chen, J. Chen, X. Jia, B. Somers, J. Wu, and P. Coppin, "A quantitative analysis of virtual endmembers' increased impact on the collinearity effect in spectral unmixing," *IEEE Trans. Geosci. Remote Sens.*, vol. 49, no. 8, pp. 2945–2956, 2011.

[15] J. M. Bioucas-Dias, A. Plaza, N. Dobigeon, M. Parente, Q. Du, P. Gader, and J. Chanussot, "Hyperspectral unmixing overview: Geometrical, statistical, and sparse regression-based approaches," *IEEE J. Sel. Topics Appl. Earth Observ.*, vol. 5, no. 2, pp. 354–379, 2012.

[16] J. B. Greer, "Sparse demixing of hyperspectral images," *IEEE Trans. Image Process.*, vol. 21, no. 1, pp. 219–228, 2012.

[17] M. Iordache, "A sparse regression approach to hyperspectral unmixing," *Universidade Técnica de Lisboa, PhD thesis*, 2011.

[18] M. Elad, "Sparse and redundant representations," *Springer*, 2010.

[19] J. M. Bioucas-Dias and M. A. Figueiredo, "Alternating direction algorithms for constrained sparse regression: Application to hyperspectral unmixing," *IEEE WHISPERS*, pp. 1–4, 2010.

[20] M. V. Afonso, J. M. Bioucas-Dias, and M. A. Figueiredo, "An augmented lagrangian approach to the constrained optimization formulation of imaging inverse problems," *IEEE Trans. Image Process.*, vol. 20, no. 3, pp. 681–695, 2011.

[21] M. Afonso, J. Bioucas-Dias, and M. Figueiredo, "A fast algorithm for the constrained formulation of compressive image reconstruction and other linear inverse problems," *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, vol. 1, pp. 4034–4037, 2010.

[22] M. Figueiredo, J. Bioucas-Dias, and M. Afonso, "Fast frame-based image deconvolution using variable splitting and constrained optimization," *Proc. IEEE Workshop Stat. Signal Process.*, vol. 1, pp. 109–112, 2009.

[23] D. C. Heinz and C. Chang, "Fully constrained least-squares linear spectral mixture analysis method for material quantification in hyperspectral imagery," *IEEE Trans. Geosci. Remote Sens*, vol. 39, no. 3, pp. 529–545, 2001.

[24] M. Winter, "Fast autonomous spectral end-member determination in hyperspectral data," *Proc.* 13^{th} *Int. Conf. Appl. Geologic Remote Sens.*, vol. 2, pp. 337–344, 1999.

[25] C.-C. Wu, S. Chu, and C.-I. Chang, "Sequential N-FINDR algorithms," *Proc. SPIE Imaging Spectrometry XIII*, vol. 7086, 2008.

[26] C.-I. Chang, C.-C. Wu, and C.-T. Tsai, "Random N-finder endmember extraction algorithms for hyperspectral imagery," *IEEE Trans. Image Process.*, vol. 20, no. 3, pp. 641-656, 2011.

[27] Y. E. Shimabukuro, A. Smith, "The least-squares mixing models to generate fraction images derived from remote sensing multispectral data," *IEEE Trans. Geosci. Remote Sens.*, vol. 29, no. 1, pp. 16-20, 1991.

[28] R. N. Clark, G. A. Swayze, R. Wise, E. Livo, T. Hoefen, R. Kokaly, and S. J. Sutley, "USGS digital spectral library splib06a", *US Geological Survey Denver*, 2007.