INTERACTIVE GAUSSIAN-SUM FILTERING FOR ESTIMATING SYSTEMATIC RISK IN FINANCIAL ECONOMETRICS

Arash Mohammadi†, Xiao-Ping Zhang⊤, and Konstantinos N. Plataniotis‡

† Concordia Institute for Information Systems Engineering, Concordia University, Montreal, QC, Canada
⊤ Electrical and Computer Engineering, Ryerson University, Toronto, ON, Canada
‡ Electrical and Computer Engineering, University of Toronto, Toronto, ON, Canada

ABSTRACT

Motivated by application of signal processing techniques in financial econometrics, we propose a novel adaptive and robust non-linear filtering methodology to estimate the systematic risk of an investment arising from exposure to general market movements. There are extensive evidence that the asset returns exhibit non-normalities due to fat tails, excessive kurtosis, and asymmetry of the financial data. Gaussian sum filters (GSF) are attractive estimators for state-estimation in such non-Gaussian problems. However, due to the high computational cost of the GSF when applied to non-Gaussian risk estimation problems, no GSF has yet been investigated in the context of financial econometrics. The paper addresses this gap. To tackle this non-Gaussian problem, we introduce the interactive Gaussian sum filter (IGSF) by representing the non-systematic part of the risk in the Fama-French multi-factor model with the Gilbert-Elliott (GE) model. Due to incorporation of the GE-model, the proposed IGSF is a computationally attractive adaptive filter with an interactive multiple model (IMM) collapsing style. In other words, the number of Gaussian components is controlled utilizing a modified Bayesian learning technique, which is used to collapse the Gaussian mixture representation of the non-systematic risk into an equivalent Gaussian term at each filtering cycle.

Index Terms— Fama-French Factor Model, Gaussian-Sum Filter, Gilbert-Elliott Model, IMM Algorithm, Non-Gaussian Noise.

1. INTRODUCTION

The objective of the paper is to dynamically estimate the systematic risk in financial econometrics using the Fama-French (FF) factor model with non-normal error distributions. The non-linear nature of the financial returns and the lack of an efficient non-Gaussian state-estimation algorithm with affordable complexity for such applications, motivate this paper. Since the 2008 financial crises which is the worst crises since the Great Depression of the 1930s, a significant focus has been given to the measurement of the systematic risk of an investment which cannot be eliminated through diversification. Recently, there has also been a surge of interest in applying signal processing tools for risk modeling/forcasting problems in financial applications [1–6]. In particular, we are interested in application of the estimation and tracking theory using factor models [7–10] for such risk management problems [11].

In the context of financial econometrics, the most well-known measure of the systematic risk is referred to as "Beta", which is a measure of the risk arising from exposure to general market movements. The most commonly models used in practice to represent Beta are the single-factor Capital Asset Pricing Model (CAPM) [12], and the multi-factor FF-model [13]. Although the CAPM model is still widely used, empirical tests show that a single factor model is not sufficient to quantify the systematic risk, leading to significant degradation in the achievable performance of the estimation algorithm. Conventionally, state-estimation algorithms represent the asset returns with a constant Beta which provides an estimate of the average risk in relation to the risk of the overall market [14]. The Ordinary Least Squares (OLS) regression is, therefore, classically used to estimate the constant Beta both in the CAPM [15, 16] and the FF [9] frameworks. Although assuming a constant Beta is papular in the financial literature, it is questionable in practice and comes with a great downfall, i.e., data frequencies and time intervals are ignored. As a result, much attention has recently been devoted to develop and use time-varying beta models [17–20]. Current research is, however, broadly centered around developing dynamical state-estimation algorithms in the CAPM framework. Application of dynamical estimators to multi-factor models, specially the FF-model, is still in its infancy. The paper addresses this gap.

Contributions: The paper proposes a Gaussian sum filter (GSF) with an interacting multiple model (IMM) collapsing step obtained by modeling the observation noise with Gilbert-Elliott (GE) model. As far as we know, no one has used the multiple model collapsing methodology in this fashion and this is the first time to make a connection between GE-model and IMM estimators. The second contribution of the paper is non-Gaussian filtering of systematic risk (Beta) via the proposed interactive Gaussian sum filter (IGSF). In particular, we focus on filtering an FF-model with non-Gaussian statistics. The IGSF resolves the computational burden of the GSF, and allows for this particular class of non-Gaussian problems in the financial econometrics to have a solution. The paper applies a new signal processing technique in a new application, this is a breakthrough in both signal processing and finance.

To better position the contributions of the paper, Table 1 provides a classification of the existing estimation methodologies in the context of financial econometrics. It is observed that very limited research has been devoted to the problems belonging to Category (v), i.e., developing state-estimators for risk management with *non-Gaussian statistics*. Although several important financial algorithms are developed based on the normality assumption, due to fat tails of asset returns [21], excessive kurtosis, and asymmetry of the financial data [22], this assumption has been firmly rejected in empirical studies [26]. In fact, there are extensive evidence [27, 28] that financial returns exhibit non-normalities. Dealing with *Non-Gaussian error distributions* is, therefore, the fundamental statistical problem when developing dynamical estimators for the FF-model. The focus of the current filtering methodologies is on developing robust estimators against outliers [21] by regularizing the Kalman filter (KF). However, regularized estimators require numerical iterations

Risk Model (Observation-Model)	Beta Model (State-Model)	Statistics	Estimator
(i) CAPM	Constant	Gaussian	OLS [15, 16]
(ii) CAPM	Time-varying	Gaussian	OLS [17]- [18]
(iii) CAPM	Constant	Non-Gaussian	Quantile Estimator [14], [19]- [22]
(iv) Fama-French	Time-varying	Gaussian	KF [23–25]
(v) Fama-French	Time-varying	Non-Gaussian	Regularized KF [9]

Table 1. Classification of filtering methodologies for Systematic risk estimation.

and their performance critically depends on the convergence, accuracy and, complexity of the underlying optimization problem. An attractive alternative solution for estimating the systematic risk is the GSF [30] which outperforms the KF-based algorithms and has reduced complexity than the particle filter. Due to its exceptional properties, the GSF has found several applications in fields as diverse as target tracking [31–33], space surveillance [34], computer vision [35], and geoscience [36]. However, due to high computational cost of the GSF when applied to non-Gaussian risk estimation problems, the GSF has not yet been applied for this particular class of problems. To address this problem, we propose to incorporate the GE-model [37, 38] model to represent the non-systematic part of the risk in the FF-framework. The GE is a classical model in communication systems for describing burst error patterns in transmission channels, and it has been widely used to model the intermittent observations over communication networks. We use the GE-model in a different and intuitively pleasing way to model the non-Gaussian noise. We develop the IGSF by incorporating an intelligent Bayesian adaptation technique to collapse the resulting Gaussian sum at each cycle into an equivalent single Gaussian term. The IGSF extends [39] by performing an IMM collapsing step instead of the multiple-model collapsing step.

2. NON-GAUSSIAN MODELING IN FF-MODEL

In this section, we develop a non-Gaussian state-space model for dynamical (time-varying) estimation of Beta. The state variables are the market Beta β_k, size Beta $\beta_k^{(\mathrm{S})}$, and value Beta $\beta_k^{(\mathrm{V})}$. The FF constitutes the observation-model, and the GE forms the non-systematic risk (noise model).

A. FF-based State-Space Model

The FF-model of a portfolio or an asset is given by [13]

$$r_k - r_k^{(f)} = \alpha_k + \big(r_k^{(I)} - r_k^{(f)}\big)\beta_k + \mathcal{S}_k\beta_k^{(\mathrm{S})} + \mathcal{H}_k\beta_k^{(\mathrm{V})} + \nu_k, \qquad (1)$$

where k denotes the time index, r_k is the return of an asset, $r_k^{(f)}$ is the risk-free return, and $r_k^{(I)}$ is the return on the market portfolio. Summation $\{r_k - r_k^{(f)}\}$ is the excess return of an asset, while $\{r_k^{(I)} - r_k^{(f)}\}$ is the excess return of the market. The additional parameter α_k represents the excess expected return of the asset over the market index. Term $\mathcal{S}_k$ (small minus big (SMB)) is the average return on three small portfolios minus the average return on three big portfolios. Term $\mathcal{H}_k$ (high minus low (HML)) is the average return on two value portfolios minus the average return on two growth portfolios [13]. Eq. (1) is used as the observation model, represented in the conventional form as

$$z_k = \boldsymbol{H}_k \boldsymbol{x}_k + \nu_k, \qquad (2)$$

where based on Eq. (1), the observation is defined as $z_k = r_k - r_k^{(f)}$, the observation model is defined as $\boldsymbol{H}_k = [1, (r_k^{(I)} - r_k^{(f)}), \mathcal{S}_k, \mathcal{H}_k]$, and the state vector is given by $\boldsymbol{x}_k = [\alpha_k, \beta_k, \beta_k^{(\mathrm{S})}, \beta_k^{(\mathrm{V})}]^T$. We formulate the state model as

$$\boldsymbol{x}_k = \boldsymbol{\Phi}\, \boldsymbol{x}_{k-1} + \begin{bmatrix} (1-\phi^{(\alpha)})\bar{\alpha} \\ (1-\phi^{(\beta)})\bar{\beta} \\ (1-\phi^{(\mathrm{S})})\bar{\beta}^{(\mathrm{S})} \\ (1-\phi^{(\mathrm{V})})\bar{\beta}^{(\mathrm{V})} \end{bmatrix} + \begin{bmatrix} \xi_k^{(\alpha)} \\ \xi_k^{(\beta)} \\ \xi_k^{(\mathrm{S})} \\ \xi_k^{(\mathrm{V})} \end{bmatrix}, \qquad (3)$$

where $\boldsymbol{\Phi}$ is a diagonal matrix of the reverting factors, i.e., $\boldsymbol{\Phi} = \mathrm{diag}(\phi^{(\alpha)}, \phi^{(\beta)}, \phi^{(\mathrm{S})}, \phi^{(\mathrm{V})})$. Terms $\bar{\beta}$, $\bar{\beta}^{(\mathrm{S})}$, and $\bar{\beta}^{(\mathrm{V})}$ are the constant means of the three betas. The state model given by Eq. (3) can accommodate different time-varying Betas depending on the values of the underlying factors: (i) *Random Coefficient Model (RCM)*: where $\phi = \phi^{(\mathrm{S})} = \phi^{(\mathrm{V})} = 0$. The RCM assumes that each beta fluctuates randomly around a mean value; (ii) *Random Walk Model (RWM)*: where $\phi = \phi^{(\mathrm{S})} = \phi^{(\mathrm{V})} = 1$, and $\bar{\beta} = \bar{\beta}^{(\mathrm{S})} = \bar{\beta}^{(\mathrm{V})} = 0$, and; (iii) *Mean Reverting Model (MRM)*, where the reverting factors and constant means are non-zero. Next, we model the observation noise ν_k in Eq. (2).

B. Gilbert-Elliott as Non-Systematic Risk Model

Term ν_k in the FF-model is the part of the return to the asset which can not be modeled and is typically referred to as the non-systematic risk. As stated previously, we propose to model the non-systematic risk ν_k by GE-mode, which is able to account for the non-normality of the Beta in the FF-framework. The GE-model is a 2-state Markov-chain with a "good" (G) and a "bad" (B) state. In the context of risk management, we model the good state (G) with a Gaussian distribution $p^{(1)}(\nu_k) = \mathcal{N}(\nu^{(1)}, R^{(1)})$ which is the probability distribution function (PDF) of the nominal/background systematic risk. The bad state (B) is modeled with $p^{(2)}(\nu_k)$ which is the PDF of the dominant heavy-tailed Gaussian noise. In this paper, we consider a Gaussian density $p^{(2)}(\nu_k) = \mathcal{N}(\nu^{(2)}, R^{(2)})$ with a large variance/covariance, i.e., ($R^{(2)} > R^{(1)}$), to model B. The proposed model serves as an approximation to the fundamental Middleton Class-A noise model which has been used extensively to model physical noise arising in radio and acoustic channels. The observation noise ν_k is completely defined by the mode transition governed by the following homogeneous Markov chain

$$\boldsymbol{P}^{(\mathrm{GE})} = \begin{bmatrix} 1-g & g \\ b & 1-b \end{bmatrix}, \qquad (4)$$

where $P(m_k^{(G)} | m_{k-1}^{(B)}) = g$ is the transition probability from state B to state G, and $P(m_k^{(B)} | m_{k-1}^{(G)}) = b$ is the transition probability in the reverse direction, from state G to state B. Note that $m_k^{(G)} \triangleq \{m_k = G\}$ is the event that mode G is in effect at iteration k. Eq. (4) models a discrete stochastic process and governs the observation noise jumps when it switches from the good state to the bad sate and vise-versa. The value of the Eq. (4) for time-varying Beta estimation is that the fat-tails in the FF-framework can be explicitly included through regime jumps. Besides, using Eq. (4), IMM-based algorithms can be developed to filter the systematic risk. The hidden Markov chain underlying the GE-model which represents the probability transition between the two Gaussian components, differentiates it from the ϵ-contaminated models. This new class of GE-based models have the ability to describe a non-Gaussian noise environment in different practical applications, such as target tracking in the presence of glint noise, spread-spectrum communication systems, and outlier rejection in image processing applications.

3. DYNAMICAL ESTIMATION OF NON-GAUSSIAN BETA

As a result of using the GE-model, we are dealing with a hybrid system with two behavioral modes (good state G and bad state B). However, in contrary to the conventional multiple-model adaptive estimation algorithms [40], both behavioural modes share the same state-model (Eq. (3)). On the other hand, the parameters of the observation model (Eq. (3)) differs from one to another, i.e., one mode uses the distribution $p^{(1)}(\nu_k)$ of the good state while the other is based on the distribution $p^{(2)}(\nu_k)$ of the bad state. The modes of the hybrid system evolve according to the Markov chain (Eq. (4)) describing the GE-model. To better motivate our filtering methodology, we note two points here: First, although there are extensive evidence that the asset returns exhibit non-normalities, GM modeling has not yet been considered for representing the non-systematic risk in the FF-framework. However, in light of Wiener theory of approximation, any non-Gaussian density can be approximated with a GM. Second, there is no GSF or IMM algorithm developed for time-varying state estimation for asset management based on factor models as the complexity of such algorithms assumed to become exhaustive over time in the financial literature. The IGSF through a modified Bayesian adaptation which incorporates the transition probability of the underlying GE-model, obtains the optimal (in the minimum mean square error (MSE) sense) single Gaussian approximation of the posterior distribution. Incorporation of GE-model in the collapsing step results in a totally novel algorithm with constant complexity.

A. The IGSF Filter with Parallel Filters

In order to estimate the state vector, one approach is to implement an IMM algorithm with two parallel filters: (i) A KF matched to the good state with $p^{(1)}(\nu_k)$ as the non-systematic risk (observation noise), and; (ii) A second KF matched to the bad state with $p^{(2)}(\nu_k)$ as the non-systematic risk. The IMM algorithm recursively runs these two filters in parallel and computes the overall estimate by forming a weighted average of the two state estimates. Each cycle of the IMM algorithm consists of four major steps: (i) Interaction, where the previous estimate of all filters are mixed together to be used as the initial conditions for the current iteration; (ii) Filtering, where m_c KFs run in parallel and form intermediate state-estimates; (iii) Mode-probability evaluation, where the probability of each behaviour mode is computed based on the innovation sequence of the individual filters, and; (iv) Combination step to compute the overall state estimate by forming a weighted average of the local state estimates. Next, we propose an alternative solution which instead uses an internal IMM-style step to collapse the Gaussian mixture representation of the predictive observation into a single Gaussian density. This allows a single Kalman recursion to be implemented at each iteration.

B. The IGSF Filter with a Single Coupled Filter

Given the record of measurements $\boldsymbol{Z}_k = [\boldsymbol{z}_1^T, \dots \boldsymbol{z}_k^T]^T = \{\boldsymbol{Z}_{k-1}, \boldsymbol{z}_k\}$, the goal of the IGSF is to compute the mean-squared-error (MSE) of the filtered estimate $\hat{\boldsymbol{x}}_{k|k} = \mathbb{E}\{\boldsymbol{x}_k|\boldsymbol{Z}_k\}$ of the state vector $\boldsymbol{x}_k$. Intuitively speaking, the IGSF approximates the predicted measurement density $f(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{Z}_{k-1})$ with a Gaussian mixture (GM) consisting of ($m_c = 2$) components by implementing m_c interconnected and coupled KFs. First we note that, modeling the non-systematic risk with the GE-model results in a GM representation for predicted measurement density. In other words, $f(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{Z}_{k-1})$ is expressed as a linear combination of ($m_c = 2$) Gaussian terms, one corresponds to the density of the good state while the other one corresponds to the density of the bad state. The predicted measurement is

$$f(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{Z}_{k-1}) = \sum_{l=1}^{m_c} f(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{Z}_{k-1}, \mathcal{F}_k^{(l)}) p(\mathcal{F}_k^{(l)}|z_k, \boldsymbol{Z}_{k-1}). \tag{5}$$

The overall state estimate at each iteration is then computed by collapsing this Gaussian sum into a single Gaussian using an IMM-type adaptive Bayesian law. In this regard, the paper establishes a close relationship between the IMM algorithm and collapsing Gaussian sums. This observation is intuitively pleasing in nature and can provide new insights into further developments in this context. The main difference between the proposed IGSF and the IMM algorithms is incorporation of the Gaussian mixture collapsing step. Specifically, there is a difference in terms of how the weights are calculated in the IGSF in comparison to the IMM algorithms, as will be covered later. Computation of (5) requires the following two terms:

1. Mode-Matched likelihood Functions: The observation likelihood conditioned on $\mathcal{F}_k^{(l)}$ (the first term on the right hand side (RHS) of Eq. (5)), is a Gaussian density as follows

$$f(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{Z}_{k-1}, \mathcal{F}_k^{(l)}) = \mathcal{N}(\hat{\boldsymbol{z}}_{k|k-1}^{(l)}, \boldsymbol{S}_k^{(l)}), \tag{6}$$

where local predictive measurement $\hat{z}_k^{(l)}$ and its associated local conditional covariance $\boldsymbol{S}_{k|k}^{(l)}$ are computed through the Kalman filter matched to each mode as follows

$$\hat{\boldsymbol{z}}_{k|k-1}^{(l)} = \boldsymbol{H}_k \hat{\boldsymbol{x}}_{k|k-1} + \nu_k^{(l)}, \tag{7}$$

$$\text{and} \quad \boldsymbol{S}_k^{(l)} = \boldsymbol{H}_k \boldsymbol{P}_{k|k-1} \boldsymbol{H}_k^T + \boldsymbol{R}_k^{(l)}. \tag{8}$$

A Gaussian sum like the one in Eq. (5) can not be easily accommodated in a recursive Kalman like filter. Thus, it has to be replaced by an equivalent Gaussian term. By applying the smoothing property of the conditional expectation operator on the predicted measurements (resulting from each member of the GE-model), we obtain the best mean-square approximation of the Gaussian sum as a single Gaussian, i.e., $f(\boldsymbol{z}_k|\boldsymbol{x}_k, \boldsymbol{Z}_{k-1}) \approx \mathcal{N}(\hat{\boldsymbol{z}}_{k|k-1}, \boldsymbol{S}_k)$, where

$$\begin{aligned} \hat{\boldsymbol{z}}_{k|k-1} &= E\{z_k|\boldsymbol{Z}_{k-1}\} = E\{E\{z_k|\mathcal{F}_k^{(l)}\}|\boldsymbol{Z}_{k-1}\} \\ &= \sum_{i=1}^{m_c} \hat{\boldsymbol{z}}_{k|k-1}^{(l)} p(\mathcal{F}_k^{(l)}|z_k, \boldsymbol{x}_k, \boldsymbol{Z}_{k-1}). \end{aligned} \tag{9}$$

Similarly, the error covariance is computed as

$$\begin{aligned} \boldsymbol{S}_k &= E\{(z_k - \hat{\boldsymbol{z}}_{k|k-1})(z_k - \hat{\boldsymbol{z}}_{k|k-1})^T|\boldsymbol{Z}_{k-1}\} \\ &= \sum_{l=1}^{m_c} w_k^{(l)} \big[\boldsymbol{S}_k^{(l)} + (\hat{\boldsymbol{z}}_{k|k-1}^{(l)} - \hat{\boldsymbol{z}}_{k|k-1})(\hat{\boldsymbol{z}}_{k|k-1}^{(l)} - \hat{\boldsymbol{z}}_{k|k-1})^T\big]. \end{aligned} \tag{10}$$

2. Adaptive Weights: We develop the weight update step of the IGSF by establishing an intuitively pleasing connection between the IGSF and the IMM algorithm. More specifically, the m_c Gaussian components in the IGSF are analogous to m_c behaviour modes of the IMM, therefore, $w_k^{(l)}$ which is the weight for the lth component of the IGSF is analogous to the probability of behaviour mode $\mathcal{F}^{(l)}$ in IMM which is computed assuming that mode $\mathcal{F}^{(l)}$ describes the true system behavior at iteration k. In the IGSF, the weights of each component is updated via a modified Bayesian law. The modified Bayesian law is derived using the GE-model and the observation innovation sequence, and by applying the Bayes' rule [30]. More specifically, the weight $w_k^{(l)}$ corresponding to the lth component can be interpreted as the conditional density $p(\mathcal{F}_k^{(l)}|\boldsymbol{Z}_k)$ corresponding to component $\mathcal{F}_k^{(l)}$, defined as

$$w_k^{(l)} \triangleq p(\mathcal{F}_k^{(l)}|\boldsymbol{Z}_k) = \frac{f(\boldsymbol{z}_k|\mathcal{F}_k^{(l)}, \boldsymbol{Z}_{k-1})p(\mathcal{F}_k^{(l)}|\boldsymbol{Z}_{k-1})}{p(\boldsymbol{z}_k|\boldsymbol{Z}_{k-1})}. \quad (11)$$

Noting that

$$p(\mathcal{F}_k^{(l)}|\boldsymbol{Z}_{k-1}) = \sum_{j=1}^{n_f} p(\mathcal{F}_k^{(l)}|\mathcal{F}_{k-1}^{(j)})p(\mathcal{F}_{k-1}^{(j)}|\boldsymbol{Z}_{k-1}), \quad (12)$$

$$\text{and } p(\boldsymbol{z}_k|\boldsymbol{Z}_{k-1}) = \sum_{j=1}^{n_f} p(\boldsymbol{z}_k|\mathcal{F}_k^{(l)}, \boldsymbol{Z}_{k-1})p(\mathcal{F}_k^{(l)}|\boldsymbol{Z}_{k-1}). \quad (13)$$

Eq. (11) can be expressed as follows

$$w_k^{(l)} = \frac{f(\boldsymbol{z}_k|\mathcal{F}_k^{(l)}, \boldsymbol{Z}_{k-1}) \sum_{j=1}^{n_f} p(\mathcal{F}_k^{(l)}|\mathcal{F}_{k-1}^{(j)})p(\mathcal{F}_{k-1}^{(j)}|\boldsymbol{Z}_{k-1})}{\sum_{j=1}^{n_f} p(\boldsymbol{z}_k|\mathcal{F}_k^{(j)}, \boldsymbol{Z}_{k-1})p(\mathcal{F}_k^{(j)}|\boldsymbol{Z}_{k-1})}. \quad (14)$$

In order to compute Eq. (14), three terms are required: (i) Term $p(\mathcal{F}_k^{(l)}|\mathcal{F}_{k-1}^{(j)})$ which is provided by the transition matrix of the GE-model; (ii) Term $p(\boldsymbol{z}_k|\boldsymbol{Z}_{k-1}, \mathcal{F}^{(l)}) = \mathcal{N}(\boldsymbol{r}_k^{(l)}, \boldsymbol{S}_k^{(l)})$ which is the augmented likelihood function of component $\mathcal{F}^{(l)}$, where $\boldsymbol{r}_k^{(l)} = \boldsymbol{z}_k - \hat{\boldsymbol{z}}_{k|k-1}^{(l)}$ is the innovation vector (obtained from Eq. (7)) and $\boldsymbol{S}_k^{(l)}$ is defined in Eq. (8), and; (iii) Term $p(\mathcal{F}_{k-1}^{(l)}|\boldsymbol{Z}_{k-1})$ reflects the prior knowledge regarding the weights before the new measurement $\boldsymbol{z}_k$ becomes available. The point estimates ($\hat{\boldsymbol{x}}_{k|k}$ and $\boldsymbol{P}_{k|k}$) are computed by forming a weighted sum of those m_c local Gaussian filters.

4. SIMULATIONS

Empirical tests are performed on real historical data from 100 randomly selected stock returns to evaluate the performance of the proposed IGSF. The stock/market returns and other required parameters are obtained from the center for research in security prices (CRSP) database, and the Fama-French datasets. As the focus of the paper is on application of GSFs for modeling of the observation noise, a single RWM state model is used in all the implemented algorithms. The stock returns from the first 120 months are used as training data to form the required parameters and the next 60 months are used as the test data to evaluate the performances. The initial state value $\boldsymbol{x}_0$, and the constant means of the state variables are computed using an OLS algorithm. The parameters of the state-space model, i.e., the covariance of the state noise $\boldsymbol{Q}$ and the observation noise variances of the GE-model, are computed in advance using maximum likelihood algorithm [39]. The coefficient of variation of the root mean squared error, CV(RMSE), is used to evaluate the performance of the proposed IGSF algorithm. The CV(RMSE) is computed as follows CV(RMSE) $= \sqrt{\frac{1}{T}\sum_{k=1}^{T}\left(r_k - \hat{r}_k\right)^2}/\bar{r}$, where $\hat{r}_k$ is the estimated asset return, and $\bar{r}$ is the mean of the asset return. Fig. 1 plots the average CV(RMSE) results computed based on three algorithms: (i) Kalman filter with Gaussian observation noise; (ii) The IMM filter, and; (iii) The proposed IGSF. It is observed that the proposed IGSF provides superior performance over its counterparts. Averaged over time and all stocks, the CV(RMSE) of the IGSF is reduced by 18.11% in comparison to the KF. This significant improvement shows the potential application of GSFs and multiple-model signal processing techniques for time-varying state estimation in financial econometrics. Multiple-model representation of the state evolution and collapsing a Gaussian sum model of the predictive distribution will be the focus of our future work to further improve the performance of the IGSF.

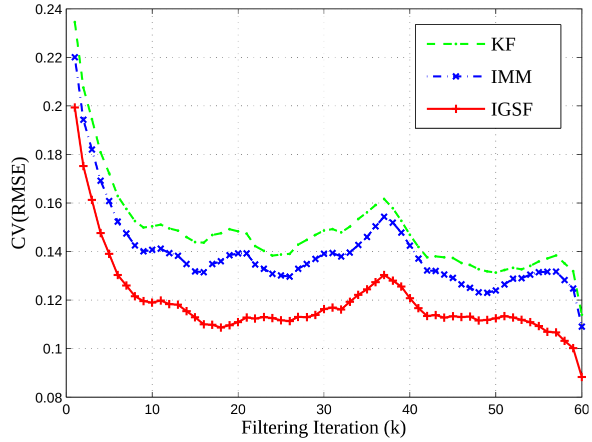

Fig. 1. The CV(RMSE) comparison between the KF; IMM filter, and; the IGSF. The filtering iterations are corresponding to monthly returns.

5. CONCLUSION

Motivated by the fact that asset returns exhibit extensive non-normalities in the context of financial econometrics, we propose the IGSF which is a GSF with an IMM collapsing step. The IGSF is derived by modeling the observation noise with GE-model. This is the first time to make a connection between GE-model and the IMM estimators. The IGSF utilizes a self-learning mechanism to resolve the computational burden of the GSF, and allows for this particular class of non-Gaussian problems in the financial econometrics to have a solution.

6. REFERENCES

[1] D.E. Johnston, I. Urteaga, P.M. Djuric, "Replication and Optimization of Hedge Fund Risk Factor Exposures," IEEE Int. Conf. Acoustics, Speech & Signal Process. (ICASSP), pp. 8712-8716, pp. 26-31, 2013.

[2] K. Slavakis, G. Leus, G.B. Giannakis, "Online Robust Portfolio Risk Management Using Total Least-Squares and Parallel Splitting Algorithms," *IEEE Int. Conf. Acoustics, Speech & Signal Process. (ICASSP)*, pp. 5686-5690, 2013.

[3] P.M. Djuric, M. Khan, D.E. Johnston, "Particle Filtering of

Stochastic Volatility Modeled With Leverage," *IEEE J. Sel. Topics Signal Process.*, vol. 6, no. 4, pp. 327-336, 2012.

[4] R. Cont, "Statistical Modeling of High Frequency Financial Data: Facts, Models, and Challenges," *IEEE Signal Process. Mag.*, vol. 28, no. 5, pp. 16-25, 2011.

[5] G. Ganesan, "A Subspace Approach to Portfolio Analysis," *IEEE Signal Process. Mag.*, vol. 28, no. 5, pp. 49-60, 2011.

[6] D.S. Matteson and D. Ruppert, "GARCH Models of Dynamic Volatility and Correlation," *IEEE Signal Process. Mag.*, vol. 28, no. 5, pp. 72-82, 2011.

[7] M. Agrawal, D. Mohapatra, and I. Pollak, "Empirical Evidence Against CAPM: Relating Alphas and Returns to Betas," *IEEE J. Sel. Topics Signal Process.*, vol. 6, no. 4, pp. 298-310, 2012.

[8] L. Vo, X.P. Zhang, and F. Wang, "Multifactor Systematic Risk Analysis Based on Piecewise Mean Reverting Model," *2013 IEEE Global Conference on Signal and Information Processing (GlobalSIP)*, 2013.

[9] E. Jay, P. Duvaut, S. Darolles, and A. Chretien, "Multifactor Models," *IEEE Signal Process. Mag.*, vol. 28, no. 5, pp. 37-48, 2011.

[10] S. Darolles, P. Duvaut, E. Jay, "Multi-factor Models and Signal Processing Techniques: Application to Quantitative Finance," *Wiley, Hoboken*, NJ, 2013.

[11] D. Johnston and P.M. Djuricc, "The Science Behind Risk Management: A Signal Processing Perspective," *IEEE Signal Process. Mag.*, vol. 28, no. 5, pp. 26-36, 2011.

[12] E.F. Fama and K.R. French, "Common risk factors in the returns on stocks and bonds," *J. Financ. Econ.*, vol. 33, no. 1, pp. 3-56, 1993.

[13] E. Fama and K. French, "Multifactor Explanations of Asset Pricing Anomalies," *J. of Financ.*, vol. 51, no. 1, 1996.

[14] A.B. Atkins and P.T. Ng "Refining Our Understanding of Beta Through Quantile Regressions," *J. Risk Financial Manag.*, vol. 7, pp. 67-79, 2014.

[15] R.D. Brooks, R.W. Faff, and J.H.H. Lee, "Beta Stability and Portfolio Formation," *Pacific Basin Financ. J.*, vol. 2, pp. 463-479, 1994.

[16] D. Kim, "The Extent of Non-stationarity of Beta," *Review of Quantitative Finance and Accounting*, pp. 241-254, 1993.

[17] P.A.V.B. Swamy and G.S. Tavlas, "Random Coefficient Models: Theory and Applications," *J. of Economic Surveys*, vol. 9, no. 2, pp. 165-196, 1995.

[18] W.F.M. De Bondt and R.H. Thaler, "A Mean-Reverting Walk Down Wall Street," *J. of Economic Perspectives*, vol. 3, no. 1, pp. 189-202, 1989.

[19] G. Joao, L. Kogan, and L. Zhang, "Equilibrium Cross-Section of Returns," *J. of Political Economy*, vol. 111, no. 4, pp. 693-732, 2003.

[20] A. Ang and J. Chen, "CAPM Over the Long-Run: 1926-2001," *J. of Empirical Financ.*, vol 14, no. 1, pp 1-40, 2007.

[21] R. Kan, and G. Zhou, "Modeling Non-Normality Using Multivariate t: Implications for Asset Pricing," *Working paper, J. M. Olin School of Bus.*, Washington University in St. Louis, 2006.

[22] M.C. Beaulieu, J.M. Dufour, and L. Khalaf, "Exact Multivariate Tests of Asset Pricing Models with Stable Asymmetric Distributions," *Numerical Methods in Financ.*, pp 173-191, 2005

[23] T. Roncalli and J. Teiletche, "An Alternative Approach to Alternative beta," *J. Financ. Transf.*, vol. 24, pp. 43-52, 2008.

[24] A. Harvey and S. J. Koopman, "Unobserved Components Models in Economics and Finance," *IEEE Control Syst. Mag.*, vol. 29, pp. 71-81, 2009.

[25] F. E. Racicot, "Modeling Hedge Funds Returns using the Kalman filter: An errors-in-variables perspective," *Int. Atlantic Economic Conf.*, 2009.

[26] J. Affeck-Graves, and B. McDonald, "Non-normalities and Tests of Asset Pricing Theories," *J. of Financ.*, vol. 44, 889-908, 1989.

[27] G. Zhou, "Asset-Pricing Tests Under Alternative Distributions," *J. Financ.*, vol. 48, pp. 1927-1942, 1993.

[28] M.C. Beaulieu, J.M. Dufour, and L. Khalaf, "Multivariate Tests of Mean-Variance Efficiency With Possibly Non-Gaussian Errors: An Exact Simulation-Based Approach," *J. of Business & Economic Statistics*, vol. 25, no. 4, pp 398-410, 2007.

[29] A.K. Gupta, T. Varga, and T. Bodnar, "Mixtures of Normal Distributions," *Elliptically Contoured Models in Statistics and Portfolio Theory*, pp 103-123.

[30] D. L. Alspach and H. W. Sorenson, "Nonlinear Bayesian Estimation Using Gaussian Sum Approximation," *IEEE Trans. Autom. Control*, vol. AC-17, pp. 439-448, 1972.

[31] A.Y. Aravkin, J.V. Burke, and G. Pillonetto, "Optimization Viewpoint on Kalman Smoothing, with Applications to Robust and Sparse Estimation," *Compressed Sensing and Sparse Filtering*, A.Y. Carmi, L.S. Mihaylova, S.J. Godsill Eds., Signals & Communications Technology, Springer, pp. 237-281, 2014.

[32] Z. Chen, "Efficient Multi-Target Tracking using Graphical Models," Thesis, Massachusetts Institute of Technology, 2008.

[33] W.I. Tam, K.N. Plataniotis, D. Hatzinakos, "An Adaptive Gaussian Sum Algorithm for Radar Tracking," *Signal Process.*, vol. 77, issue 1, pp. 85-104, 1999.

[34] J.T. Horwood and A.B. Poore, "Adaptive Gaussian Sum Filters for Space Surveillance," *IEEE Trans. Autom. Control*, vol. 56, no. 8, pp. 1777-1790, 2011.

[35] B. Han, D. Comaniciu, Y. Zhu, and L. Davis, "Sequential Kernel Density Approximation and Its Application to Real-Time Visual Tracking," *IEEE Trans. Pattern Anal. Mach. Intell.*, pp. 227-234, 2008.

[36] K. Vishwajeet and P. Singla, "Nonlinear Uncertainty Propagation for Perturbed Two-Body Orbits," *J. Guidancce, Cont., & Dynamics*, 2014.

[37] K. Liu, Q. Zhao, and B. Krishnamachari, "Dynamic Multichannel Access With Imperfect Channel State Detection," *IEEE Trans. Signal Process.*, vol. 58, pp. 2795-2808, 2010.

[38] A.J. Goldsmith and P.P. Varaiya, "Capacity, mutual information, and coding for finite-state Markov channels," *IEEE Trans. Inf. Theory*, vol. 42, no. 3, pp. 868-886, 1996.

[39] K.N. Plataniotis, D. Androutsos, A.N. Venetsanopoulos, "Nonlinear Filtering of Non-Gaussian Noise", *J. of Intelligent and Robot. Syst.*, vol. 19. pp. 207-231,1997.

[40] X.R. Li and V.P. Jilkov, V.P., "Survey of maneuvering target tracking. Part V. Multiple-model methods," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 41, no. 4, pp. 1255-1321, 2005.