Solutions 10

Exercise 2.9

a) According to the Bayes risk function

$$R(\delta,\pi_1) = C_{00}(1-P_F(\delta))+C_{10}P_F(\delta)+\pi_1((C_{11}-C_{00})+(C_{01}-C_{11})P_M(\delta)-(C_{10}-C_{00})P_F(\delta)), \tag{1}$$

we have following equation

$$\begin{cases} R(\delta,0) = P_F(\delta) \\ R(\delta,1) = 2-2P_D(\delta). \end{cases} \tag{2}$$

Then, the minimax equation is given by

$$P_F = 2(1-P_D). \tag{3}$$

Combine the conclusion of Problem 2.8, we search for the minimax test δ_M.

$$\begin{cases} P_F = 2-2P_D \\ P_D = -P_F^2+2P_F \end{cases} \Longrightarrow \begin{cases} P_F = \frac{1}{2} \\ P_D = \frac{3}{4} \end{cases} \Longrightarrow \tau_M = 1. \tag{4}$$

According to the test, we have

$$\pi_{0M} = \left[1+\frac{1}{2}\right]^{-1} = \frac{2}{3}. \tag{5}$$

b) For the given costs, the Bayes risk function can also be expressed as follows:

$$R(\delta,\pi_0) = 2(1-\pi_0)(1-P_D(\delta)) + \pi_0 P_F(\delta). \tag{6}$$

Firstly, we need notice that the $P_F = P_D = 0$ if $\tau > 2$.

- 1) When $\tau > 2$:
  In this case, $\pi_0 > \frac{4}{5}$, and the optimum Bayesian risk function $V(\pi_0)$ is

$$V(\pi_0) = 2(1-\pi_0). \tag{7}$$

- 2) When $\tau \leq 2$:

In this case, $\pi_0 \leq \frac{4}{5}$, and the Bayesian risk function is

$$R(\delta, \pi_0) = \frac{1-\pi_0}{2}\tau^2 - \frac{\pi_0}{2}\tau + \pi_0 = \frac{1-\pi_0}{2}\left(\tau - \frac{\pi_0}{2(1-\pi_0)}\right)^2 + \frac{8\pi_0 - 9\pi_0^2}{8(1-\pi_0)}. \quad (8)$$

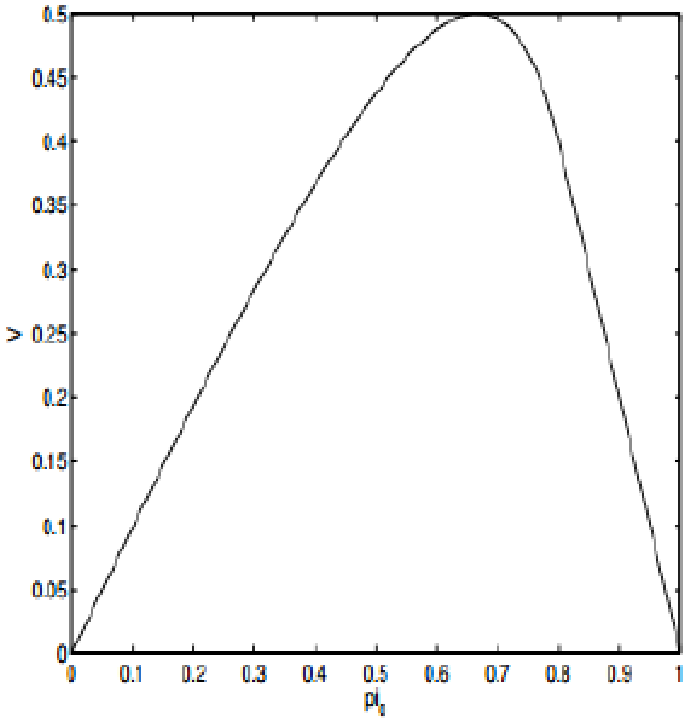
Therefore, the optimum Bayesian risk function $V(\pi_0)$ is

$$V(\pi_0) = \frac{8\pi_0 - 9\pi_0^2}{8(1-\pi_0)}. \quad (9)$$

In a word, the optimum Bayesian risk function can be expressed as

$$V(\pi_0) = \begin{cases} 2 - 2\pi_0 & \pi_0 > \frac{4}{5} \\ \frac{8\pi_0 - 9\pi_0^2}{8(1-\pi_0)} & \pi_0 \leq \frac{4}{5}. \end{cases} \quad (10)$$

Plot the $V(\pi_0)$:



Exercise 2.11

It is same with Exercise 2.9.

Exercise 2.12

a) Since the known additive constant A shifts the distribution of Y by A, the densities of Y under H_0 and H_1 are give respectively by

$$f(y|H_0) = \begin{cases} 1 & -\frac{1}{2} \leq y \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

$$f(y|H_1) = \begin{cases} 1 & -\frac{1}{2} + A \leq y \leq \frac{1}{2} + A \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Accordingly, the likelihood ratio function can take 3 possible values:

$$L(y) = \begin{cases} 0 & -\frac{1}{2} \leq y \leq -\frac{1}{2} + A \\ 1 & -\frac{1}{2} + A \leq y \leq \frac{1}{2} \\ \infty & \frac{1}{2} \leq y \leq \frac{1}{2} + A \end{cases} . \tag{13}$$

There are three cases:

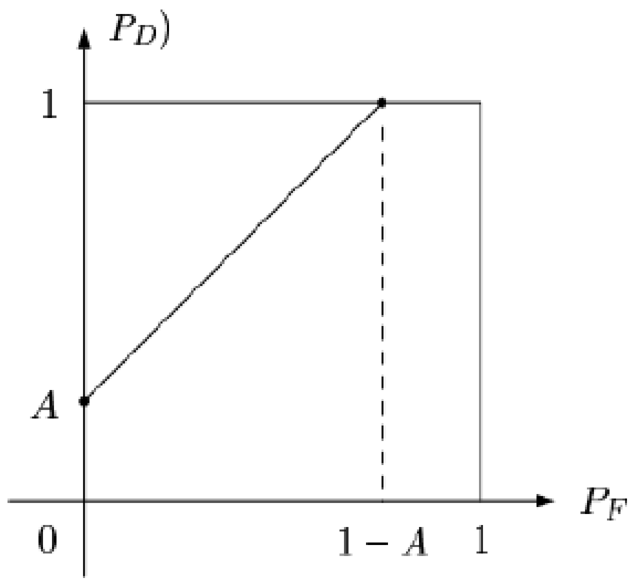
- 1) $\tau < 0$:
  In this case, $P_D = P_F = 1$.
- 2) $0 < \tau < 1$:
  In this case, $P_D = 1$, and P_F can be calculated as

$$P_F = \int_{A-\frac{1}{2}}^{\frac{1}{2}} 1 dy = 1 - A. \tag{14}$$

- 3) $1 < \tau < \infty$:
  In this case, $P_F = 0$, and P_D can be calculated as

$$P_D = \int_{\frac{1}{2}}^{A+\frac{1}{2}} 1 dy = A. \tag{15}$$

Then, we plot the ROC as



According to the ROC, we can know there need a randomization if $\alpha \in (A, 1)$ with probability as

$$p = \frac{1-(1-\alpha)}{1-A} = \frac{\alpha}{1-A}. \tag{16}$$

b) The Bayes risk function:

$$R = \pi_0 P_F + 2(1-\pi_0)(1-P_D). \tag{17}$$

Then, we derive the optimum Bayes risk function:

$$V(\pi_0) = \begin{cases} \frac{1}{2}\pi_0 & \pi_0 < \frac{2}{3} \\ 1-\pi_0 & \pi_0 > \frac{2}{3} \end{cases}. \tag{18}$$

And the minimax equation is

$$P_F = 2 - 2P_D \tag{19}$$

Then, we derive

$$\begin{cases} P_D = \frac{5}{6} \\ P_F = \frac{1}{3} \end{cases}. \tag{20}$$

Here, we need randomization with probability $p = \frac{2}{3}$.

Exercise 2.2.12

Part 1 The conditional Bayes Risk is

$$R(\delta|H_0) = C_{10}P_F(\delta) + C_{00}(1-P_F(\delta)) \tag{21}$$

and

$$R(\delta|H_1) = C_{11}P_D(\delta) + C_{01}(1-P_D(\delta)) \tag{22}$$

Hence the Bayes Risk is

$$\begin{aligned} R(\delta,\phi(\mathbf{R})) &= [C_{10}P_F(\delta) + C_{00}(1-P_F(\delta))](1-\phi(\mathbf{R})) \\ &\quad + [C_{11}P_D(\delta) + C_{01}(1-P_D(\delta))]\phi(\mathbf{R}) \\ &= C_{00}(1-\phi(\mathbf{R})) + C_{01}\phi(\mathbf{R}) \\ &\quad + (C_{10}-C_{00})P_F(\delta)(1-\phi(\mathbf{R})) + (C_{11}-C_{01})P_D(\delta)\phi(\mathbf{R}), \end{aligned} \tag{23}$$

where $P_F(\delta) = \sum_{y\in\mathcal{Y}_1} P(y|H_0)$ and $P_D(\delta) = \sum_{y\in\mathcal{Y}_1} P(y|H_1)$.

Part 2 The LRT that minimize the Bayes Risk is $L(\delta,\phi(\mathbf{R})) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$, where

$$\tau = \frac{(C_{10}-C_{00})(1-\phi(\mathbf{R}))}{(C_{11}-C_{01})\phi(\mathbf{R})}. \tag{24}$$

It is a specific decision rule, and the case that $(\delta,\phi(\mathbf{R})) = \tau$ doesn't influence the Bayes Risk since $(C_{10}-C_{00})P_F(\delta)(1-\phi(\mathbf{R})) + (C_{11}-C_{01})P_D(\delta)\phi(\mathbf{R}) = 0$. Hence a randomized test is not necessary.

Part 3 The slope of and straight-line segment on the ROC is a constant, which is exactly the LRT. For the same LRT, we have the same decision rule hence $P_F(\delta)$ and $P_D(\delta)$ are constant. Then the Bayes Risk is constant.

Part 4 If P_F is continuous function with respect to τ, then $P_F = \alpha$. If it is a discrete function, then we need to find p that

$$p(1-\alpha^+) + (1-p)(1-\alpha^-) = 1-\alpha \tag{25}$$

and hence $p = \frac{\alpha^- - \alpha}{\alpha^- - \alpha^+}$. Then $\phi(\mathbf{R}) = \frac{\alpha^- - \alpha}{\alpha^- - \alpha^+}$.

Exercise 2.2.15

Part 1

$$\begin{aligned}
\sqrt{2\pi}\text{erfc}(x) &= \int_x^\infty e^{-\frac{t^2}{2}}dt \\
&= \frac{1}{x}e^{-\frac{x^2}{2}} - \int_x^\infty \frac{1}{t^2}e^{-\frac{t^2}{2}}dt \\
&= f_1(x) \\
&= \frac{1}{x}e^{-\frac{x^2}{2}} - \frac{1}{x^3}e^{-\frac{x^2}{2}} + \int_x^\infty \frac{3}{t^4}e^{-\frac{t^2}{2}}dt \\
&= f_2(x)
\end{aligned} \tag{26}$$

We know

$$f_1(x) \le \frac{1}{x}e^{-\frac{x^2}{2}} \tag{27}$$

and

$$\begin{aligned}
f_2(x) &\ge \frac{1}{x}e^{-\frac{x^2}{2}} - \frac{1}{x^3}e^{-\frac{x^2}{2}} \\
&= \frac{1}{x}e^{-\frac{x^2}{2}}\left(1 - \frac{1}{x^2}\right)
\end{aligned} \tag{28}$$

Hence we finish the proof.

Part 2

Solution 1 By mathematical induction we can finish the proof.

Suppose

$$\text{erfc}(x) = \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}\left[1 + \sum_{m=1}^{n-1}(-1)^m\frac{1\cdot 3\cdots(2m-1)}{x^{2m}} + R_n\right] \tag{29}$$

and

$$R_n = (-1)^n x e^{\frac{x^2}{2}}\int_x^{+\infty}\frac{1\cdot 3\cdots(2n-1)}{t^{2n}}e^{-\frac{t^2}{2}}dt \tag{30}$$

Firstly, from part 1 we have

$$\begin{aligned}
\operatorname{erfc}(x) &= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \left[1 - x e^{\frac{x^2}{2}} \int_x^{+\infty} \frac{1}{t^2} e^{-\frac{t^2}{2}} dt \right] \\
&= \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \left[1 + R_1 \right] \\
&= \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \left[1 - \frac{1}{x^2} + x e^{\frac{x^2}{2}} \int_x^{\infty} \frac{3}{t^4} e^{-\frac{t^2}{2}} dt \right] \\
&= \frac{1}{\sqrt{2\pi}x} e^{-\frac{x^2}{2}} \left[1 - \frac{1}{x^2} + R_2 \right]
\end{aligned} \tag{31}$$

Then the assumption holds for $n = 1$ and $n = 2$. Suppose that it holds for the k-th item, then for $k + 1$-th term we have

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \left(1 + \sum_{m=1}^{k-1} (-1)^m \frac{1 \cdot 3 \cdots (2m-1)}{x^{2m}} + R_k \right) \tag{32}$$

where

$$\begin{aligned}
R_k &= (-1)^k x e^{\frac{x^2}{2}} \int_x^{+\infty} \frac{1 \cdot 3 \cdots (2k-1)}{t^{2k}} e^{-\frac{t^2}{2}} dt \\
&= (-1)^k x e^{\frac{x^2}{2}} \left[\frac{1 \cdot 3 \cdots (2k-1)}{x^{2k+1}} e^{-\frac{x^2}{2}} - \int_x^{+\infty} \frac{1 \cdot 3 \cdots (2k-1)(2k+1)}{t^{2k+2}} e^{-\frac{t^2}{2}} dt \right] \\
&= (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}} + (-1)^{k+1} x e^{\frac{x^2}{2}} \int_x^{+\infty} \frac{1 \cdot 3 \cdots (2k-1)(2k+1)}{t^{2k+2}} e^{-\frac{t^2}{2}} dt \\
&= (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}} + R_{k+1}
\end{aligned} \tag{33}$$

Hence

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \left(1 + \sum_{m=1}^{k} (-1)^m \frac{1 \cdot 3 \cdots (2m-1)}{x^{2m}} + R_{k+1} \right) \tag{34}$$

and the assumption is right.

From (33) we can get R_k has the same sign with $(-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}}$ when $x > 0$, i.e. R_k has the same sign as the $k + 1$-th term. Notice that R_{k+1} has the different sign from R_k, then $|R_k| < \frac{1 \cdot 3 \cdots (2k-1)}{x^{2k}}$, i.e. the remainder is less than the magnitude of the $k + 1$-th term.

Solution 2

$$
\begin{aligned}
\mathsf{erfc}(x) &= \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}} - \int_x^\infty e^{-\frac{u^2}{2}}\frac{1}{u^2}du \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}} - \frac{1}{x^3}e^{-\frac{x^2}{2}} + \int_x^\infty \frac{3}{u^4}e^{-\frac{u^2}{2}}du \\
&\vdots \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}\left[1+\sum_{m=1}^{n-1}(-1)^m\frac{1\cdot 3\cdot\dots\cdot(2m-1)}{x^{2m}}\right] \\
&\quad + \frac{1}{\sqrt{2\pi}}(-1)^n 1\cdot 3\cdot\dots\cdot(2n-1)\int_x^\infty \frac{1}{u^{2n}}e^{-\frac{u^2}{2}}du
\end{aligned}
\tag{35}
$$

If we want to derive R_n, we need prove that

$$
\int_x^\infty \frac{1}{u^{2n}}e^{-\frac{u^2}{2}}du = \frac{1}{x}e^{-\frac{x^2}{2}}\frac{1}{x^{2n}}\theta \tag{36}
$$

And the proof is:

$$
\begin{aligned}
\int_x^\infty \frac{1}{u^{2n}}e^{-\frac{u^2}{2}}du &\overset{t=\frac{u^2-x^2}{2}}{\Longrightarrow} \int_0^\infty \frac{1}{(2t+x^2)^n}e^{-(t+\frac{x^2}{2})}\frac{1}{\sqrt{2t+x^2}}dt \\
&= e^{-\frac{x^2}{2}}\cdot\frac{1}{x^{2n+1}}\int_0^\infty e^{-t}(1+\frac{1}{x^2})^{-n-\frac{1}{2}}dt \\
&= e^{-\frac{x^2}{2}}\cdot\frac{1}{x^{2n+1}}\theta
\end{aligned}
\tag{37}
$$

Therefore,

$$
\mathsf{erfc}(x) = \frac{1}{\sqrt{2\pi}x}e^{-\frac{x^2}{2}}\left[1+\sum_{m=1}^{n-1}(-1)^m\frac{1\cdot 3\cdot\dots\cdot(2m-1)}{x^{2m}}+R_n\right] \tag{38}
$$

where

$$
R_n = \left[(-1)^n\cdot\frac{1\cdot 3\cdot\dots\cdot(2n-1)}{2^{2n}}\right]\int_0^\infty e^{-t}(1+\frac{2t}{x^2})^{-n-\frac{1}{2}}dt \tag{39}
$$

Exercise 2.2.17

Part 1 The LRT is

$$
\begin{aligned}
L(X_1,X_2) &= \frac{p_{x_1,x_2|H_1}(x_1,x_2|H_1)}{p_{x_1,x_2|H_0}(x_1,x_2|H_0)} \\
&= \frac{\sigma_0\left[\exp\left(\frac{X_1^2}{2\sigma_0^2}-\frac{X_1^2}{2\sigma_1^2}\right)+\exp\left(\frac{X_2^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]}{2\sigma_1}
\end{aligned}
\tag{40}
$$

Part 2 We have

$$\begin{aligned} P_D &= P(H_1|H_1) \\ &= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\ &= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \end{aligned} \tag{41}$$

and

$$\begin{aligned} P_F &= P(H_1|H_0) \\ &= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\ &= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \end{aligned} \tag{42}$$

Let $L(x_1, x_2) \geq \tau$, we have

$$\exp\left(\frac{\sigma_1^2-\sigma_0^2}{2\sigma_0^2\sigma_1^2}x_1^2\right)+\exp\left(\frac{\sigma_1^2-\sigma_0^2}{2\sigma_0^2\sigma_1^2}x_2^2\right)\geq\frac{2\tau\sigma_1}{\sigma_0} \tag{43}$$

And here we need divide the problem into 2 cases to discuss.

- 1) If $\sigma_1 > \sigma_0$, then the upper bound region is

  $$|x| \geq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{2\tau\sigma_1}{\sigma_0}-1\right)} = C_1 \tag{44}$$

  and the lower bound region is

  $$|x| \geq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{\tau\sigma_1}{\sigma_0}\right)} = C_2 \tag{45}$$

The upper bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\leq \frac{1}{\pi\sigma_1\sigma_0}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 2\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 4Q(\frac{C_1}{\sigma_1})Q(\frac{C_1}{\sigma_0})
\end{aligned} \tag{46}$$

The lower bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\geq \frac{1}{\pi\sigma_1\sigma_0}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 2\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= 4Q(\frac{C_2}{\sigma_1})Q(\frac{C_2}{\sigma_0})
\end{aligned} \tag{47}$$

The upper bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\leq \frac{1}{2\pi\sigma_0^2}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_1}^{\infty}\int_{x_2=C_1}^{\infty}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= Q^2(\frac{C_1}{\sigma_0})
\end{aligned} \tag{48}$$

The lower bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\geq \frac{1}{2\pi\sigma_0^2}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=C_2}^{\infty}\int_{x_2=C_2}^{\infty} \exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= Q^2(\frac{C_2}{\sigma_0})
\end{aligned} \tag{49}$$

- 2) If $\sigma_1 < \sigma_0$, then the upper bound region is

$$|x| \leq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{2\tau\sigma_1}{\sigma_0}-1\right)} = C_1 \tag{50}$$

and the lower bound region is

$$|x| \leq \sqrt{\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2-\sigma_0^2}\ln\left(\frac{\tau\sigma_1}{\sigma_0}\right)} = C_2 \tag{51}$$

The upper bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\leq \frac{1}{4\pi\sigma_1\sigma_0}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \left(1-2Q(\frac{C_1}{\sigma_1})\right)\left(1-2Q(\frac{C_1}{\sigma_0})\right)
\end{aligned} \tag{52}$$

The lower bound of P_D:

$$\begin{aligned}
P_D &= P(H_1|H_1) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_1}(x_1,x_2|H_1)dx_1dx_2 \\
&= \frac{1}{4\pi\sigma_1\sigma_0}\int_{L(x_1,x_2)\geq\tau}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&\leq \frac{1}{4\pi\sigma_1\sigma_0}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi}\sigma_1}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2}\left[\exp\left(-\frac{X_1^2}{2\sigma_1^2}-\frac{X_2^2}{2\sigma_0^2}\right)+\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_1^2}\right)\right]dx_1dx_2 \\
&= \left(1-2Q(\frac{C_2}{\sigma_1})\right)\left(1-2Q(\frac{C_2}{\sigma_0})\right)
\end{aligned} \tag{53}$$

The upper bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\leq \frac{1}{2\pi\sigma_0^2}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_1}^{C_1}\int_{x_2=-C_1}^{C_1}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \left(1-2Q(\frac{C_1}{\sigma_0})\right)^2
\end{aligned} \tag{54}$$

The lower bound of P_F:

$$\begin{aligned}
P_F &= P(H_1|H_0) \\
&= \int_{L(x_1,x_2)\geq\tau} p_{x_1,x_2|H_0}(x_1,x_2|H_0)dx_1dx_2 \\
&= \frac{1}{2\pi\sigma_0^2}\int_{L(x_1,x_2)\geq\tau}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&\leq \frac{1}{2\pi\sigma_0^2}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \frac{1}{\sqrt{2\pi}\sigma_0}\frac{1}{\sqrt{2\pi}\sigma_0}\int_{x_1=-C_2}^{C_2}\int_{x_2=-C_2}^{C_2}\exp\left(-\frac{X_1^2}{2\sigma_0^2}-\frac{X_2^2}{2\sigma_0^2}\right)dx_1dx_2 \\
&= \left(1-2Q(\frac{C_2}{\sigma_0})\right)^2
\end{aligned} \tag{55}$$

Exercise 2.2.19

Part 1 The LRT is

$$
\begin{aligned}
L(R) &= \prod_{i=1}^{N} \frac{p_{r_i|H_1}(R_i|H_1)}{p_{r_i|H_0}(R_i|H_0)} \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N \prod_{i=1}^{N} \exp\left[\frac{(R_i-m_0)^2}{2\sigma_0^2} - \frac{(R_i-m_1)^2}{2\sigma_1^2}\right] \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N \prod_{i=1}^{N} \exp\left[\frac{\sigma_1^2(R_i-m_0)^2 - \sigma_0^2(R_i-m_1)^2}{2\sigma_0^2\sigma_1^2}\right] \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N \prod_{i=1}^{N} \exp\left[\frac{(\sigma_1^2-\sigma_0^2)R_i^2 - 2R_i(\sigma_1^2 m_0 - \sigma_0^2 m_1) + \sigma_1^2 m_0^2 - \sigma_0^2 m_1^2}{2\sigma_0^2\sigma_1^2}\right] \\
&= \left(\frac{\sigma_0}{\sigma_1}\right)^N \exp\left[\frac{(\sigma_1^2-\sigma_0^2)l_\beta - 2l_\alpha(\sigma_1^2 m_0 - \sigma_0^2 m_1) + N\sigma_1^2 m_0^2 - N\sigma_0^2 m_1^2}{2\sigma_0^2\sigma_1^2}\right]
\end{aligned}
\tag{56}
$$

Part 2 If $2m_0 = m_1$, $2\sigma_1 = \sigma_0$, we can derive the LRT is

$$
\begin{aligned}
L(R) = 2^N \mathsf{exp}\left(\frac{-3l_\beta + 14l_\alpha m_0 - 15Nm_0^2}{8\sigma_1^2}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \tau \\
\Longrightarrow l_\beta \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{8}{3}\sigma_1^2(\ln\tau - N\ln 2) + \frac{14}{3} l_\alpha m_0 - 5Nm_0^2.
\end{aligned}
\tag{57}
$$

Therefore, there is a line separate the l_α, l_β-plane to two parts. The slope and the intercept of the line are

$$
\begin{cases}
k = \frac{14}{3} m_0 \\
b = -\frac{8}{3}(\ln\tau - N\ln 2)\sigma_1^2 - 5Nm_0^2
\end{cases}
\tag{58}
$$

The region above the line is H_0 and the other is H_1.