

Solutions 2

Exercise 3.10

$$\begin{aligned} \mathbf{I}(\boldsymbol{\theta}) &= \mathbb{E} \left[\frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}} \right] \\ \mathbf{a}^T \mathbf{I}(\boldsymbol{\theta}) \mathbf{a} &= \mathbb{E} \left[\mathbf{a}^T \frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \cdot \frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})^T}{\partial \boldsymbol{\theta}} \mathbf{a} \right] \\ &= \mathbb{E} \left[\left(\mathbf{a}^T \frac{\partial \ln p(\mathbf{x}; \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right)^2 \right] \geq 0 \end{aligned}$$

for any \mathbf{a} and all $\boldsymbol{\theta}$, thus it is positive semi-definite. For Problem 3.3, letting $\boldsymbol{\theta} = [A \ r]^T$ and using (3.31) in Page 47, we can get

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = \frac{1}{\sigma^2} \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i} \right]^T \left[\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j} \right]$$

where $\boldsymbol{\mu}(\boldsymbol{\theta}) = [A \ Ar \ \dots \ Ar^{N-1}]$. Thus,

$$\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial A} = [1 \ r \ \dots \ r^{N-1}]^T \quad \frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial r} = A [0 \ 1 \ \dots \ (N-1)r^{N-2}]^T$$

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} \sum_{n=0}^{N-1} r^{2n} & A \sum_{n=0}^{N-1} nr^{2n-1} \\ A \sum_{n=0}^{N-1} nr^{2n-1} & A^2 \sum_{n=0}^{N-1} n^2 r^{2n-2} \end{bmatrix}$$

If $A = 0$, $\mathbf{I}(\boldsymbol{\theta})$ is not positive definite.

Exercise 3.14

Since $x[n] \sim \mathcal{N}(A, \sigma_A^2)$, we can get $\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$.

$$\mathbb{E}(\hat{A}|A = A_0) = A_0 \quad \text{var}(\hat{A}|A = A_0) = \frac{\sigma^2}{N} \rightarrow 0, N \rightarrow \infty$$

So $\hat{A} \rightarrow A_0$ when $N \rightarrow \infty$. Consider the estimator $\hat{\sigma}_A^2 = (\hat{A})^2$, we have $\mathbb{E}[\hat{\sigma}_A^2] = \mathbb{E}[\hat{A}^2] =$

$\text{var}(\hat{A}) + \mathbb{E}[\hat{A}]^2 = A_0^2 + \frac{\sigma^2}{N} \rightarrow A_0^2$ when $N \rightarrow \infty$. Since $\mathbb{E}[A^2] = \text{var}[A] = \sigma_A^2$ and

$$\begin{aligned}\mathbb{E}[A^4] &= \int_{-\infty}^{\infty} A^4 \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{A^2}{2\sigma_A^2}} dA \\ &= \frac{1}{\sqrt{2\pi\sigma_A^2}} \int_{-\infty}^{\infty} A^3 (-\sigma_A^2) de^{-\frac{A^2}{2\sigma_A^2}} \\ &= \frac{1}{\sqrt{2\pi\sigma_A^2}} \cdot \left[A^3 (-\sigma_A^2) e^{-\frac{A^2}{2\sigma_A^2}} \Big|_{-\infty}^{\infty} + 3 \int_{-\infty}^{\infty} A^2 \sigma_A^2 e^{-\frac{A^2}{2\sigma_A^2}} dA \right] \\ &= 3\sigma_A^2 \int_{-\infty}^{\infty} A^2 \frac{1}{\sqrt{2\pi\sigma_A^2}} e^{-\frac{A^2}{2\sigma_A^2}} dA \\ &= 3\sigma_A^4\end{aligned}$$

we can get $\text{var}(\hat{\sigma}_A^2) = \text{var}(\hat{A}^2) = \mathbb{E}[A^4] - \mathbb{E}[A^2]^2 = 2\sigma_A^4$. So that $\text{var}(\hat{\sigma}_A^2) \rightarrow 2\sigma_A^4$, which is just the CRLB as $N \rightarrow \infty$. The $\hat{\sigma}_A^2$ cannot be estimated without error because we cannot reduce the random nature of A when we only have one realization of A .

Exercise 3.15

Since $x[n]$ s are independent, we have $\mathbf{I}(\rho) = N\mathbf{i}(\rho)$, where $\mathbf{i}(\rho)$ is the Fisher information matrix for a single vector sample. Using (3.32), we can get

$$\mathbf{i}(\rho) = \frac{1}{2} \text{tr} \left[\left(\mathbf{C}^{-1}(\rho) \frac{\partial \mathbf{C}(\rho)}{\partial \rho} \right) \right]$$

and

$$\mathbf{C}^{-1}(\rho) = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \quad \frac{\partial \mathbf{C}(\rho)}{\partial \rho} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus,

$$\mathbf{i}(\rho) = \frac{1+\rho^2}{(1-\rho^2)^2} \quad \mathbf{I}(\rho) = \frac{N(1+\rho^2)}{(1-\rho^2)^2}$$

We can get CRLB for ρ as $\text{var}(\rho) \geq \frac{(1-\rho^2)^2}{N(1+\rho^2)}$.

Exercise 3.17

Similar to example 3.14 in Page 56, we can get Fisher information matrix over the interval $n = -M, \dots, 0, \dots, M$ except $[\mathbf{I}(\boldsymbol{\theta})]_{23} = [\mathbf{I}(\boldsymbol{\theta})]_{32} = \frac{\pi A^2}{\sigma^2} \sum_{n=-M}^{n=M} n = 0$.

$$\mathbf{I}(\boldsymbol{\theta}) = \frac{1}{\sigma^2} \begin{bmatrix} \frac{2M+1}{2} & 0 & 0 \\ 0 & 2A^2\pi^2 \sum_{n=-M}^M n^2 & 0 \\ 0 & 0 & \frac{(2M+1)A^2}{2} \end{bmatrix}$$

Thus, the CRLB is

$$\text{var}(\hat{A}) \geq \frac{2\sigma^2}{2M+1} = \frac{2\sigma^2}{N} \quad \text{var}(\hat{\phi}) \geq \frac{2\sigma^2}{(2M+1)A^2} = \frac{1}{N\eta}$$

$$\text{var}(\hat{f}_0) \geq \frac{\sigma^2}{2A^2\pi^2 \sum_{n=-M}^M n^2} = \frac{6\sigma^2}{4A^2\pi^2 M(M+1)(2M+1)} = \frac{12}{(2\pi)^2 \eta N(N^2-1)}$$

where $N = 2M+1$ and $\eta = A^2/(2\sigma^2)$. The results of \hat{A} and \hat{f}_0 are the same as in example, however $\hat{\phi}$ is different.

Proof of the Asymptotic CRLB

Please see Appendix 3D in Page 77, and the details will be presented on office hours.