Solutions 3

Exercise 5.3

$$p(\mathbf{x};\lambda) = \prod_{n=0}^{N-1} \lambda \exp\left(-\lambda x[n]\right) \qquad x[n] > 0$$
$$= \lambda^N \exp\left(-\lambda \sum_{n=0}^{N-1} x[n]\right) \cdot u(\min x[n])$$

Thus, $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]$ is a sufficient statistic.

Exercise 5.4

$$p(\mathbf{x};\lambda) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(x[n]-\theta)^2}$$
$$= \frac{1}{(2\pi\theta)^{N/2}} e^{-\frac{1}{2\theta}\sum_{n=0}^{N-1}(x[n]-\theta)^2}$$
$$= \frac{1}{(2\pi\theta)^{N/2}} e^{-\frac{1}{2\theta}\sum_{n=0}^{N-1} x^2[n] - \frac{1}{2}N\theta} \cdot e^{\sum_{n=0}^{N-1} x[n]}$$

Thus, $T(\mathbf{x}) = \sum_{n=0}^{N-1} x^2[n]$ is a sufficient statistic.

Exercise 5.5

$$p(x[n];\theta) = \frac{1}{2\theta}\left(u(x[n]+\theta) - u(x[n]-\theta)\right)$$

$$p(\mathbf{x};\theta) = \frac{1}{(2\theta)^N} \prod_{n=0}^{N-1} \left(u(x[n]+\theta) - u(x[n]-\theta)\right)$$

The product is zero unless $-\theta \le x[n] \le \theta$ for all $x[n]$ or $\max|x[n]| \le \theta$, so that

$$p(\mathbf{x};\theta) = \frac{1}{(2\theta)^N} u(\theta - \max|x[n]|)$$

Thus, $T(\mathbf{x}) = \max|x[n]|$ is a sufficient statistic.

Exercise 5.14

(a)

$$p(x;\mu) = \exp\left[\mu x - \frac{1}{2}x^2 + (-\frac{1}{2}\mu^2 + \ln\frac{1}{\sqrt{2\pi}})\right]$$

where $A(\mu) = \mu$, $B(x) = x$, $C(x) = -\frac{1}{2}x^2$ and $D(\mu) = -\frac{1}{2}\mu^2 + \ln\frac{1}{\sqrt{2\pi}}$.

(b)

$$\begin{aligned} p(x;\sigma^2) &= \frac{x}{\sigma^2}\exp{[-\frac{1}{2}\frac{x^2}{\sigma^2}]}u(x) \\ &= \exp{[-\frac{1}{2}\frac{x^2}{\sigma^2} + \ln xu(x) - \ln\sigma^2]} \end{aligned}$$

where $A(\sigma^2)B(x) = -\frac{1}{2}\frac{x^2}{\sigma^2}$, $C(x) = \ln xu(x)$ and $D(\sigma^2) = -\ln\sigma^2$.

(c)

$$p(x;\lambda) = \exp{[-\lambda x + \ln u(x) + \ln\lambda]}$$

where $A(\lambda)B(x) = -\lambda x$, $C(x) = \ln u(x)$ and $D(\lambda) = \ln\lambda$.

Exercise 5.15

$$\begin{aligned} p(\mathbf{x},\theta) &= \prod_{n=0}^{N-1}\exp[A(\theta)B(x[n]) + C(x[n]) + D(\theta)] \\ &= \exp[A(\theta)\sum_n B(x[n]) + \sum_n C(x[n]) + ND(\theta)] \\ &= \exp[A(\theta)\sum_n B(x[n]) + ND(\theta)]\exp[\sum_n C(x[n])] \end{aligned}$$

Thus, $T(x) = \sum_{n=0}^{N-1} B(x[n])$ is a sufficient statistic.

For (a), $T(x) = \sum_n x[n]$, for (b), $T(x) = \sum_n x^2[n]$ and for (c), $T(x) = \sum_n x[n]$.

For Gaussian, the MVU is $\hat{\mu} = \frac{1}{N}T(x) = \frac{1}{N}\sum_n x[n]$.

For Rayleigh, we have $\mathbb{E}(x^2) = 2\sigma^2$. Thus, $\hat{\sigma^2} = \frac{1}{2N}\sum_n x^2[n]$.

For exponential, we have $\mathbb{E}(x) = \frac{1}{\lambda}$.Denote $\nu = \frac{1}{\lambda}$, then $\hat{\nu} = \frac{1}{N}\sum_n x[n]$.

Exercise 5.16

$$p(\mathbf{x};\theta) = \frac{1}{(2\pi)^{N/2}\det^{1/2}(\mathbf{C})}\exp\{-\frac{1}{2}(\mathbf{x}-\mu)^T\mathbf{C}^{-1}(\mathbf{x}-\mu)\}$$

where

$$\det(\mathbf{C}) = N - 1 + \sigma^2 - (-\mathbf{1})^T(-\mathbf{1}) = \sigma^2$$

and

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{\sigma^2} & \frac{1}{\sigma^2}\mathbf{1}^T \\ \frac{1}{\sigma^2}\mathbf{1} & \left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{N-1+\sigma^2}\right)^{-1} \end{bmatrix}$$

We can get $\left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{N-1+\sigma^2}\right)^{-1} = \mathbf{I} + \frac{\mathbf{1}\mathbf{1}^T}{\sigma^2}$

$$\mathbf{C}^{-1} = \frac{1}{\sigma^2}\begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & \sigma^2\mathbf{I} + \mathbf{1}\mathbf{1}^T \end{bmatrix}$$

$$(\mathbf{x}-\mu)^T\mathbf{C}^{-1}(\mathbf{x}-\mu) = \frac{1}{\sigma^2}[x[0]-N\mu, x[1], \ldots, x[N-1]] \begin{bmatrix} 1 & \mathbf{1}^T \\ \mathbf{1} & \sigma^2\mathbf{I}+\mathbf{1}\mathbf{1}^T \end{bmatrix} \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$= \frac{1}{\sigma^2}\left[(\sum_{n=0}^{N-1} x[n] - N\mu)^2 + \sigma^2 \sum_{n=0}^{N-1} x[n]^2\right]$$

$$= \frac{N^2}{\sigma^2}(\bar{x}-\mu)^2 + \sum_{n=0}^{N-1} x^2[n]$$

Thus,

$$p(\mathbf{x};\boldsymbol{\theta}) = \frac{1}{(2\pi)^{N/2}\sigma} \exp\left\{-\frac{1}{2}\left[\frac{N^2}{\sigma^2}(\bar{x}-\mu)^2 + \sum_{n=0}^{N-1} x^2[n]\right]\right\}$$

$\bar{x}$ is the sufficient statistic for $\boldsymbol{\theta}$.

Exercise 5.18

$$p(\mathbf{x},\boldsymbol{\theta}) = \frac{1}{(\theta_2-\theta_1)^N} u(\min x[n]-\theta_1) u(\max x[n]-\theta_2)$$

The sufficient statistic is

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} \min x[n] \\ \max x[n] \end{bmatrix}$$