Solutions 4

Exercise 6.1

Since $\mathbb{E}\{x[n]\} = Ar^n$, we have $\mathbf{H} = (1, r, \ldots, r^{N-1})^T$ and $\mathbf{C}^{-1} = \frac{1}{\sigma^2}\mathbf{I}$. Thus,

$$\hat{A} = \frac{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}}{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}} = \frac{\sum_{n=0}^{N-1} x[n]r^n}{\sum_{n=0}^{N-1} r^{2n}}$$

$$\text{var}(\hat{A}) = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1} = \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}}$$

$\text{var}(\hat{A}) \to 0$ only if $|r| \geq 1$.

Exercise 6.4

(a) For Laplacian distribution, we have $\mathbb{E}\{x[n]\} = \mu$ and $\text{var}(x[n]) = 2$. Thus, we have $\mathbf{H} = \mathbf{1}$, $\mathbf{C}^{-1} = \frac{1}{2}\mathbf{I}$ and

$$\hat{\mu} = \frac{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}}{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

(b) For Gaussian distribution, the same with (a), we have $\hat{\mu} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$.

In the Gaussian case, the BLUE is also MVU estimator, but not true in the Laplacian case.

Exercise 6.5

Let $y = \ln x$, we can get

$$\begin{aligned}
\mathbb{E}(x) &= \int_0^{\infty} \frac{x}{\sqrt{2\pi}x} \exp\left[-\frac{1}{2}(\ln x - \theta)^2 dx\right] \\
&= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(y - \theta)^2 + y\right] \\
&= \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(y - (\theta + 1))^2\right] \exp(\theta + \frac{1}{2}) \\
&= \exp(\theta + \frac{1}{2})
\end{aligned}$$

Also, $y \sim \mathcal{N}(0, 1)$. The BLUE is $\hat{\theta} = \frac{1}{N}\sum_{n=0}^{N-1} y[n] = \frac{1}{N}\sum_{n=0}^{N-1} \ln x[n]$.

Exercise 6.9

$\mathbf{H} = (1, \cos 2\pi f_1, \ldots, \cos 2\pi(N-1)f_1)^T$ and $\mathbf{C}^{-1} = \frac{1}{\sigma^2}\mathbf{I}$

$$\hat{A} = \frac{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}}{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}} = \frac{\sum_{n=0}^{N-1} x[n]\cos 2\pi f_1 n}{\sum_{n=0}^{N-1} \cos^2 2\pi f_1 n}$$

$$\mathrm{var}(\hat{A}) = (\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H})^{-1} = \frac{\sigma^2}{\sum_{n=0}^{N-1} \cos^2 2\pi f_1 n}$$

Obviously, when $\cos^2 2\pi f_1 n = 1$, we can obtain minimum $\mathrm{var}(\hat{A})$. Thus, $f_1 = 0$.

Exercise 6.14

$$\mathbb{E}\left[w[n]\right] = (1-\epsilon)\mathbb{E}\left[w_B[n]\right] + \epsilon\mathbb{E}\left[w_I[n]\right] = 0$$

$$\begin{aligned}\mathbb{E}\left[w^2[n]\right] &= \int_{-\infty}^{+\infty} w^2 p(w)dw \\ &= (1-\epsilon)\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_B^2}} e^{-\frac{w^2}{2\sigma_B^2}} dw + \epsilon\int_{-\infty}^{+\infty} w^2 \frac{1}{\sqrt{w\pi\sigma_I^2}} e^{-\frac{w^2}{2\sigma_I^2}} \\ &= (1-\epsilon)\sigma_B^2 + \epsilon\sigma_I^2\end{aligned}$$

$$\mathsf{var}(w[n]) = (1-\epsilon)\sigma_B^2 + \epsilon\sigma_I^2$$

Then, we let $y[n] = w^2[n] + z[n]$, where $z[n]$ is the noise with zero-mean, and its variance matrix is $\sigma_Z^2\mathbf{I}$. Here, $\mathbf{H} = \mathbf{1}$ and $\mathbf{C} = \sigma_Z^2\mathbf{I}$.

We can find the BLUE of σ^2 first as follows:

$$\hat{\sigma}^2 = \frac{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{x}}{\mathbf{H}^T\mathbf{C}^{-1}\mathbf{H}} = \frac{1}{N}\sum_{n=0}^{N-1} w^2[n]$$

Then, using the result in Prob. 6.12, we can get

$$\begin{aligned}\hat{\sigma_I}^2 &= \frac{\hat{\sigma}^2 - (1-\epsilon)\sigma_B^2}{\epsilon} \\ &= \frac{\frac{1}{N}\sum_{n=0}^{N-1} w^2[n] - (1-\epsilon)\sigma_B^2}{\epsilon}\end{aligned}$$

Exercise 7.1

$$\frac{\partial \ln p(\mathbf{x};A)}{\partial A} = -\frac{N}{2A} + \frac{1}{A}\sum_{n=0}^{N-1}(x[n]-A) + \frac{1}{2A^2}\sum_{n=0}^{N-1}(x[n]-A)^2$$

$$\frac{\partial \ln p^2(\mathbf{x};A)}{\partial A^2} = \frac{N}{2A^2} - \frac{1}{A^2}\sum_{n=0}^{N-1} x[n] - \frac{1}{A^2}\sum_{n=0}^{N-1}(x[n]-A) - \frac{1}{A^3}\sum_{n=0}^{N-1}(x[n]-A)^2$$

Since $\mathbb{E}\{x[n]\} = A$ and $\mathrm{var}(x[n]) = A$, we have

$$\mathbb{E}[\frac{\partial \ln p^2(\mathbf{x};A)}{\partial A^2}] = -\frac{N}{2A^2} - \frac{N}{A}$$

$$\text{var}(\hat{A}) \geq \frac{A^2}{N(A+1/2)}$$

Exercise 7.5

According to Chebychev's inequality, we have $\Pr\{|\hat{\theta}-\theta| > \epsilon\} \leq \frac{\text{var}(\hat{\theta})}{\epsilon^2}$, where $\hat{\theta} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$ and $\text{var}(\hat{\theta}) = \frac{\sigma^2}{N}$. Thus,

$$\lim_{N\to\infty} \Pr\{|\hat{\theta}-\theta| > \epsilon\} \leq \lim_{N\to\infty} \frac{\sigma^2}{\epsilon^2 N} = 0$$

Exercise 7.6

Linearizing g about the true value of θ, i.e. θ_0, yields

$$\alpha = g(\theta) \approx g(\theta_0) + \frac{dg}{d\theta}|_{\theta=\theta_0}(\theta-\theta_0)$$

$$\hat{\alpha}-\alpha = g(\hat{\theta}) - g(\theta_0) \approx [g(\theta_0) + \frac{dg}{d\theta}|_{\theta=\theta_0}(\hat{\theta}-\theta_0)] - g(\theta_0) = \frac{dg}{d\theta}|_{\theta=\theta_0}(\hat{\theta}-\theta_0)$$

$$\Pr\{|\hat{\alpha}-\alpha| > \epsilon\} = \Pr\{|\frac{dg}{d\theta}|_{\theta=\theta_0}(\hat{\theta}-\theta_0)| > \epsilon\} = \Pr\{|\hat{\theta}-\theta_0| > \frac{\epsilon}{|\frac{dg}{d\theta}|_{\theta=\theta_0}|}\}$$

Since $\hat{\theta}$ is consistent for θ, we have $\lim_{N\to\infty}\Pr\{|\hat{\theta}-\theta| > \varepsilon\} = 0$ as long as $dg/d\theta$ is bounded. Hence, $\Pr\{|\hat{\alpha}-\alpha| > \epsilon\} \to 0$ as $N \to \infty$

Exercise 7.16

$$\begin{aligned}
\text{var}(\hat{A}) &= \text{var}(x[0]) = \text{var}(w[0]) \\
&= \int_{-\infty}^{\infty} u^2 \frac{1}{2} e^{-|u|} du \\
&= \int_{0}^{\infty} u^2 e^{-u} du \\
&= 2
\end{aligned}$$

$$\frac{dp(u)}{du} = \begin{cases} -\frac{1}{2}e^{-u}, & u > 0 \\ \frac{1}{2}e^{u}, & u < 0 \end{cases}$$

$$\begin{aligned}
\text{var}(\hat{A}) &\geq \int_{-\infty}^{\infty} (\frac{dp(u)}{du})^2/p(u) du \\
&= \int_{-\infty}^{\infty} \frac{1}{2} e^{-|u|} du \\
&= 1
\end{aligned}$$

Thus, MLE doesn't attain the CRLB as $N \to \infty$.

Exercise 7.19

$$p(\mathbf{x}; f_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - \cos 2\pi f_0 n)^2\}$$

To find MLE, we should minimize

$$\begin{aligned}\sum_n (x[n] - \cos 2\pi f_0 n)^2 &= \sum_n x^2[n] - 2\sum_n x[n]\cos 2\pi f_0 n + \sum_n \cos^2 2\pi f_0 n \\ &= \sum_n x^2[n] - 2\sum_n x[n]\cos 2\pi f_0 n + \sum_n \frac{1-\cos 4\pi f_0 n}{2} \\ &\approx \sum_n x^2[n] - 2\sum_n x[n]\cos 2\pi f_0 n + \frac{N}{2}\end{aligned}$$

where $\sum_n \cos 4\pi f_0 n \approx 0$.

To maximize $\sum_n x[n]\cos 2\pi f_0 n$ over $0 < f_0 < \frac{1}{2}$, we use Newton-Raphson iteration,

$$g(f_0) = \sum_n x[n]\cos 2\pi f_0 n$$

$$dg/df_0 = -\sum_n (2\pi n)x[n]\sin 2\pi f_0 n$$

$$d^2g/df_0^2 = -\sum_n (2\pi n)^2 x[n]\cos 2\pi f_0 n$$

Thus,

$$f_0^{(k+1)} = f_0^{(k)} - \frac{dg/df_0}{d^2g/df_0^2}\Big|_{f_0=f_0^{(k)}} = f_0^{(k)} - \frac{\sum_n (2\pi n)x[n]\sin 2\pi f_0 n}{\sum_n (2\pi n)^2 x[n]\cos 2\pi f_0 n}\Big|_{f_0=f_0^{(k)}}$$

The code for grid search:

```
clear all;close all;clc;
M = 300;
N = 10; f_0 = 0.25; sigam2 = 0.01;
n = 0:1:(N-1);
n = n';
u = cos(2*pi*f_0*n);
theta = ones(N,1)*sqrt(sigam2);
f_e = 0:0.0005:0.5;
for i=1:M
    x = normrnd(u,theta); % generate data
    for j = 1:1:length(f_e)
    val_re(1,j) = x'*cos(2*pi*n*f_e(j));
end
[a,b] = max(val_re);
result_f(1,i) = f_e(1,b);
end
plot(1:1:M,result_f);
f_mean= mean(result_f)
f_var = var(result_f)
```

The code for Newton-Raphson:

```
clear all;close all;clc;
M = 300;
N = 10; f_0 = 0.25; sigam2 = 0.01;
n = 0:1:(N-1);
n = n';
u = cos(2*pi*f_0*n);
theta = ones(N,1)*sqrt(sigam2);
f_init = 0.24;   % the initial point
        can only set in the interval
        [0.23,0.27], otherwise it diverges.
for i=1:M
    x = normrnd(u,theta);  % generate data
    f_k = f_init;
    for j=1:20    %control the iteration
        ud_val = (x'*((2*pi*n).*(sin(2*pi*f_k*n))))...
            /(x'*(((2*pi*n).^2).*(cos(2*pi*f_k*n))));
        f_k = f_k - ud_val;
    end
    result_f(1,i) = f_k;
end
plot(1:1:M, result_f);
f_mean = mean(result_f)
f_var = var(result_f)
```

The code for Appendix 7A:

```
clear all;close all;clc;
M = 30000;  % M times experiment
N = 100;    % N data in each experiment
A = 0.25;   % The real A

A_m = zeros(1,M);
for i=1:M
    %% Step 1
    u1 = rand(N,1);
    u2 = rand(N,1);

    %% Step 2
    w = sqrt(A) * sqrt(-2*log(u1)).* cos(2*pi*u2);

    %% Step 3
    x = A * ones(N,1) + w;

    %% Step 4
    A_m(1,i) = -0.5 + sqrt( mean(x.^2) + (1/4));
end
[n,x]= hist(A_m,20);
m_A = mean(A_m);      % The mean
var_A = var(A_m);     % The variance
plot(x,n/M);
grid on;
title(' The pdf of the estimat A ');
```