Solutions 5

Exercise 10.1

$$
\begin{aligned}
\hat{\theta} = \mathbb{E}(\theta|\mathbf{x}) &= \int \theta p(\theta|\mathbf{x}) d\theta \\
&= \int \theta \frac{p(\mathbf{x}|\theta)p(\theta)}{p(\mathbf{x})} d\theta \\
&= \frac{\int \theta p(\mathbf{x}|\theta)p(\theta)d\theta}{\int p(\mathbf{x}|\theta)p(\theta)d\theta} \\
&= \frac{\int \theta p(\mathbf{x}|\theta)\delta(\theta-\theta_0)d\theta}{\int p(\mathbf{x}|\theta)\delta(\theta-\theta_0)d\theta} \\
&= \theta_0 \frac{p(\mathbf{x}|\theta_0)}{p(\mathbf{x}|\theta_0)} \\
&= \theta_0
\end{aligned}
$$

The MMSE estimator is just the true value since our prior knowledge is perfect.

Exercise 10.2

$$
\begin{aligned}
p_x(x[0], x[1]) &= p_w(x[0]-A, x[1]-A|A) \\
&= p_w(x[0]-A)p_w(x[1]-A) \\
&= p_x(x[0]|A)p_x(x[1]|A)
\end{aligned}
$$

Thus, $x[0]$ and $x[1]$ are conditionally independent. However, the covariance of $x[0]$ and $x[1]$ is

$$
\begin{aligned}
\mathrm{Cov}(x[0], x[1]) &= \mathbb{E}\left[(x[0]-\mathbb{E}\left[x[0]\right])(x[1]-\mathbb{E}\left[x[1]\right])\right] \\
&= \mathbb{E}\left[(A+w[0])(A+w[1])\right] \\
&= \mathbb{E}\left[A^2\right] + \mathbb{E}\left[A\right]\mathbb{E}\left[w[0]\right] + \mathbb{E}\left[A\right]\mathbb{E}\left[w[1]\right] + \mathbb{E}\left[w[0]\right]\mathbb{E}\left[w[1]\right] \\
&= 1
\end{aligned}
$$

Thus, $x[0]$ and $x[1]$ are correlated, which means they are not independent unconditionally.

Exercise 10.3

$$
p(\mathbf{x}|\theta) = e^{-\sum_n (x[n]-\theta)} u(\min x[n] - \theta)
$$

$$
\begin{aligned}
p(\theta|\mathbf{x}) &= \frac{p(\mathbf{x}|\theta)p(\theta)}{\int p(\mathbf{x}|\theta)p(\theta)} d\theta \\
&= \frac{e^{-N(\bar{x}-\theta)e^{-\theta}} u(\min x[n]-\theta)u(\theta)}{\int_0^{\min x[n]} e^{-N(\bar{x}-\theta)}e^{-\theta} d\theta} \\
&= \frac{(N-1)e^{\theta(N-1)}}{e^{(N-1)\min x[n]}-1} u(\min x[n]-\theta)u(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\theta} = \mathbb{E}(\theta|\mathbf{x}) &= \frac{\int_0^{\min x[n]} \theta(N-1)e^{\theta(N-1)} d\theta}{e^{(N-1)\min x[n]}-1} \\
&= \frac{N-1}{e^{(N-1)\min x[n]}-1} \int_0^{\min x[n]} \theta e^{\theta(N-1)} d\theta \\
&= \frac{\min x[n] e^{(N-1)\min x[n]}}{e^{(N-1)\min x[n]}-1} - \frac{1}{N-1}
\end{aligned}
$$

So,

$$
\hat{\theta} = \frac{\min x[n] \cdot e^{(N-1)\min x[n]}}{e^{(N-1)\min x[n]}-1} - \frac{1}{N-1}
$$

Exercise 10.4

$$
p(\mathbf{x}|\theta) = \frac{1}{\theta^N} u(\min x[n]) u(\theta - \max x[n])
$$

$$
p(\theta) = \frac{1}{\beta} u(\theta) u(\beta - \theta)
$$

$$
p(\theta|\mathbf{x}) = \frac{(N-1)}{\theta^N(\max x[n]^{-(N-1)} - \beta^{-(N-1)})}
$$

$$
\hat{\theta} = \frac{N-1}{N-2} \cdot \frac{(\max x[n])^{-(N-2)} - \beta^{-(N-2)}}{(\max x[n])^{-(N-1)} - \beta^{-(N-1)}}
$$

Note that for β and N large, we have

$$
\hat{\theta} \approx \frac{N-1}{N-2} \max x[n] \approx \max x[n]
$$

which agrees with the MLE.

Exercise 10.5

$$
\begin{aligned}
\text{Bmse}(\hat{\theta}) &= \mathbb{E}_{x,\theta} \left\{ [(\theta - \mathbb{E}(\theta|\mathbf{x})) + (\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})]^2 \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{\theta|x} \left\{ [(\theta - \mathbb{E}(\theta|\mathbf{x})) + (\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})]^2 \right\} \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{\theta|x}\{(\theta - \mathbb{E}(\theta|\mathbf{x}))^2\} + 2\mathbb{E}_{\theta|x}\{(\theta - \mathbb{E}(\theta|\mathbf{x}))(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})\} + \mathbb{E}_{\theta|x}\{(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})^2\} \right\} \\
&= \mathbb{E}_x \left\{ \mathbb{E}_{\theta|x}\{(\theta - \mathbb{E}(\theta|\mathbf{x}))^2\} + \mathbb{E}_{\theta|x}\{(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})^2\} \right\}
\end{aligned}
$$

To minimize $\text{Bmse}(\hat{\theta})$, is equal to minimize $\mathbb{E}_{\theta|x}\{(\mathbb{E}(\theta|\mathbf{x}) - \hat{\theta})^2\}$. Thus, $\hat{\theta} = \mathbb{E}(\theta|x)$

Exercise 10.6

$$p(x[n]|A) = \begin{cases} \frac{1}{\sqrt{2\pi\sigma_+^2}} e^{-\frac{1}{2\sigma_+^2}(x[n]-A)^2} & A \geq 0 \\ \frac{1}{\sqrt{2\pi\sigma_-^2}} e^{-\frac{1}{2\sigma_-^2}(x[n]-A)^2} & A < 0 \end{cases}$$

$$p(\mathbf{x}|A) = \begin{cases} \frac{1}{(2\pi\sigma_+^2)^{N/2}} e^{-\frac{1}{2\sigma_+^2}\sum_n (x[n]-A)^2} & A \geq 0 \\ \frac{1}{(2\pi\sigma_-^2)^{N/2}} e^{-\frac{1}{2\sigma_-^2}\sum_n (x[n]-A)^2} & A < 0 \end{cases}$$

For $\sigma_+^2 = \sigma_-^2$, $p(\mathbf{x}|A) = p(\mathbf{x}; A)$, and for $\sigma_+^2 \neq \sigma_-^2$, $p(\mathbf{x}|A) \neq p(\mathbf{x}; A)$

Exercise 10.10

$$\begin{aligned} p(\theta|\mathbf{x}) &= \frac{p(\mathbf{x}|\theta)p(\theta)}{\int p(\mathbf{x}|\theta)p(\theta)d\theta} \\ &= \frac{\theta^N e^{-\theta\sum_{n=0}^{N-1} x[n]} u(\min x[n]) \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} u(\min\theta)}{\int_{\min\theta}^{\infty} \theta^N e^{-\theta\sum_{n=0}^{N-1} x[n]} \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} d\theta} \\ &= \frac{\theta^N e^{-\theta\sum_{n=0}^{N-1} x[n]} u(\min x[n]) \frac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\lambda\theta} u(\min\theta)}{\frac{\Gamma(N+\alpha)}{(\sum_{n=0}^{N-1} x[n]+\lambda)^{(N+\alpha)}} \int_{\min\theta}^{\infty} \theta^{N+\alpha-1} e^{-\theta(\sum_{n=0}^{N-1} x[n]+\lambda)} \frac{(\sum_{n=0}^{N-1} x[n]+\lambda)^{(N+\alpha)}}{\Gamma(N+\alpha)} d\theta} \\ &= \frac{(\sum_{n=0}^{N-1} x[n]+\lambda)^{(N+\alpha)} \theta^{N+\alpha-1} e^{-\theta(\sum_{n=0}^{N-1} x[n]+\lambda)} u(\min x[n]) u(\min\theta)}{\Gamma(N+\alpha)} \end{aligned}$$

Therefore, $p(\theta|\mathbf{x})$ has the same form with $p(\theta)$.

Exercise 10.11

From Fig. 10.9a, we can see the PDF of the random vector $[h, w]^T$ is correlated within an ellipse. This could be a Gaussian PDF. Hence, we estimate w based on h using a MMSE estimation from (10.16)

$$\hat{w} = \mathbb{E}(w) + \frac{\mathrm{Cov}(h, w)}{\mathrm{var}(h)}(h - \mathbb{E}(h))$$

From Fig. 10.9b, there does not appear to be any correlation between height and weight. Thus, $\hat{w} = \mathbb{E}(w) = 150$ for all height.