Detection & Estimation Lecture 4

Bayesian

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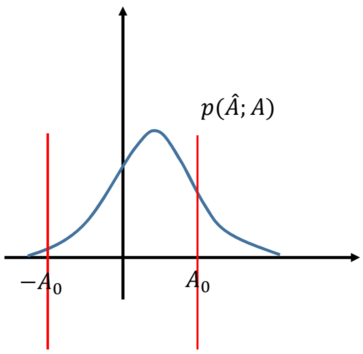
Prior Knowledge

- Physical constraint: $A \in [-A_0, A_0]$



$$\hat{A} = \frac{1}{N}\sum x[n]$$

$$\check{A} = \begin{cases} -A_0, & \hat{A} < -A_0 \\ \hat{A}, & \hat{A} \in [-A_0, A_0] \\ A_0, & \hat{A} > A_0 \end{cases}$$

$$mse(\hat{A}) > mse(\check{A})$$

Prior Knowledge

- Knowing that the unknown parameter must lie in a known interval, we may assign a uniform PDF and model the true value as a realization of a random variable
- Bayesian MSE

$$Bmse(\hat{A}) = E_{A,\boldsymbol{x}}\left[(A - \hat{A})^2\right]$$

Bmse vs MSE

$$Bmse(\hat{A}) = \iint (A - \hat{A})^2 p(x, A)\,dx\,dA$$

$$mse(\hat{A}) = \int (\hat{A} - A)^2 p(x; A)\,dx$$

Minimum Bmse Estimator

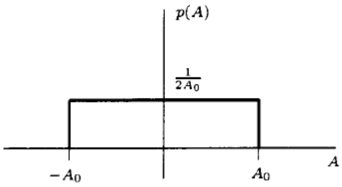
$$Bmse(\hat{A}) = \int \left[\int (A - \hat{A})^2 p(A|x) dA \right] p(x) dx$$

minimize

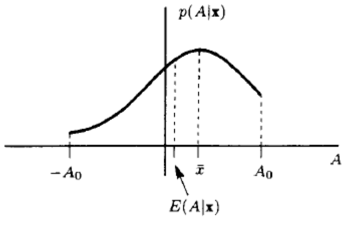
$$\hat{A} = E[A|x] = \int A p(A|x) dA$$

Posterior vs Prior PDF

$$p(A) = \int p(x, A) dx$$

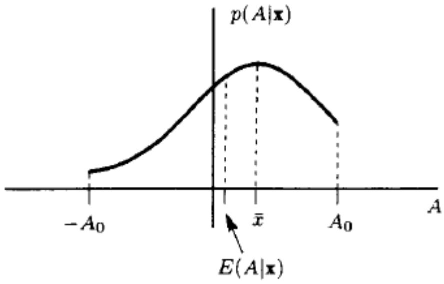


$$p(A|x) = \frac{p(x, A)}{p(x)}$$



Posterior PDF

$$p(A|\mathbf{x}) = \begin{cases} \dfrac{\dfrac{1}{2A_0(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\dfrac{1}{2\sigma^2}\displaystyle\sum_{n=0}^{N-1}(x[n]-A)^2\right]}{\displaystyle\int_{-A_0}^{A_0} \dfrac{1}{2A_0(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\dfrac{1}{2\sigma^2}\displaystyle\sum_{n=0}^{N-1}(x[n]-A)^2\right] dA} & |A| \leq A_0 \\ 0 & |A| > A_0. \end{cases}$$



Gaussian Prior

$$p(A) = \begin{cases} \dfrac{1}{2A_0} & |A| \leq A_0 \\ 0 & |A| > A_0 \end{cases}$$ uniform prior

$$p(A) = \frac{1}{\sqrt{2\pi\sigma_A^2}} \exp\left[-\frac{1}{2\sigma_A^2}(A-\mu_A)^2\right]$$ Gaussian prior

Gaussian Prior

$$p(A|\mathbf{x}) = \frac{\exp\left[-\frac{1}{2\sigma^2_{A|x}}(A-\mu_{A|x})^2\right]\exp\left[-\frac{1}{2}\left(\frac{\mu_A^2}{\sigma_A^2}-\frac{\mu^2_{A|x}}{\sigma^2_{A|x}}\right)\right]}{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma^2_{A|x}}(A-\mu_{A|x})^2\right]\exp\left[-\frac{1}{2}\left(\frac{\mu_A^2}{\sigma_A^2}-\frac{\mu^2_{A|x}}{\sigma^2_{A|x}}\right)\right]dA}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2_{A|x}}}\exp\left[-\frac{1}{2\sigma^2_{A|x}}(A-\mu_{A|x})^2\right]$$

$$\hat{A} = \frac{\sigma_A^2}{\sigma_A^2+\frac{\sigma^2}{N}}\bar{x} + \frac{\frac{\sigma^2}{N}}{\sigma_A^2+\frac{\sigma^2}{N}}\mu_A$$

$$= \alpha\bar{x} + (1-\alpha)\mu_A$$

$$\alpha = \frac{\sigma_A^2}{\sigma_A^2+\frac{\sigma^2}{N}}$$

Gaussian PDF Properties

Theorem 10.1 (Conditional PDF of Bivariate Gaussian) *If x and y are distributed according to a bivariate Gaussian PDF with mean vector $[E(x)\,E(y)]^T$ and covariance matrix*

$$\mathbf{C} = \begin{bmatrix} \operatorname{var}(x) & \operatorname{cov}(x,y) \\ \operatorname{cov}(y,x) & \operatorname{var}(y) \end{bmatrix}$$

so that

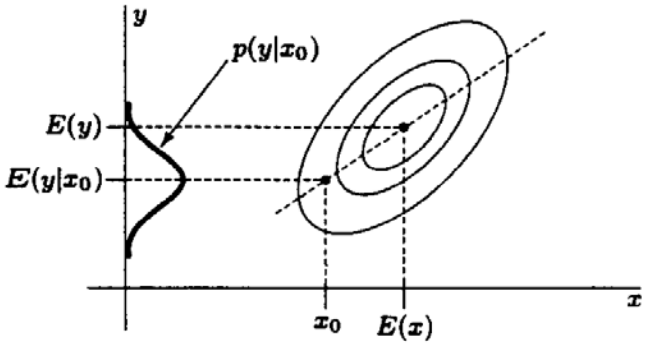
$$p(x,y) = \frac{1}{2\pi \det^{\frac{1}{2}}(\mathbf{C})}\exp\left[-\frac{1}{2}\begin{bmatrix} x-E(x) \\ y-E(y) \end{bmatrix}^T \mathbf{C}^{-1}\begin{bmatrix} x-E(x) \\ y-E(y) \end{bmatrix}\right],$$

then the conditional PDF $p(y|x)$ is also Gaussian and

$$E(y|x) = E(y) + \frac{\operatorname{cov}(x,y)}{\operatorname{var}(x)}(x-E(x))$$

$$\operatorname{var}(y|x) = \operatorname{var}(y) - \frac{\operatorname{cov}^2(x,y)}{\operatorname{var}(x)}.$$

Gaussian PDF Properties



HW

- Chapter 10
  - 10.1, 10.2, 10.3, 10.4, 10.5, 10.6, 10.10, 10.11