Proof of the Number of Independent Kirchhoff Equations in an Electrical Circuit

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***Abstract*—This brief paper presents a compact inductive proof that the number of linearly independent KCL node equations of a b-branch n-node connected circuits is $n-1$ and the number of independent KVL loop equations is $b-n+1$. Because it is easily illustrated pictorially and does not rely on graph theoretic concepts it is suitable for presentation at an elementary level of circuit theoretic instruction.**

IN the many popular elementary circuits books, the numbers of linearly independent Kirchhoff's current and voltage law equations are stated either without proof or with proofs based on graph theoretic concepts such as fundamental cutsets and loops [1]–[8]. These proofs are usually found in dedicated graph theoretic chapters which typically are skipped when the book is used for an introductory course [9]. The reason is that they can introduce a great deal of excess nomenclature not very useful for most students who ultimately may master only nodal and mesh analysis. In [10] and [11] the authors guide the reader through a succession of problems toward a proof. However, the proofs are incomplete, because the number of independent KVL based equations is proven for planar circuits only. Even earlier books written before the introduction of graph theory in circuit analysis do not offer formal proofs [12]–[15]. The inductive proof to be presented here is both complete and elementary in that it is intuitive and easily illustrated with simple pictures.

First we need some definitions. A circuit is an interconnection of electrical components, i.e., resistances, capacitances, inductances, sources, etc. These electrical components form the *branches* of the circuit. Two or more branches are connected together at the *nodes* of the circuit. A *path* in the circuit is an enumeration of branches where any two successive branches are adjacent. A path having the same starting and ending node forms a *loop*. A circuit is considered to be *connected* if there is a path of branches between any pair of nodes in the circuit.

A set of equations is said to be *linearly independent* if none of them can be obtained from the others by performing a sequence of linear operations, i.e., adding, subtracting equations, or multiplying them by a constant [16].

Theorem: Let a connected circuit have n nodes and b branches. There are exactly $n-1$ linearly independent equations based on Kirchhoff's current law (KCL) at the nodes and $b-n+1$ linearly independent equations based on Kirchhoff's voltage law (KVL) around loops of the circuit.

Observe that in the theorem formulation we refer only to equations based on KCL at the nodes of the circuit. The theorem and its demonstration can naturally be extended to general KCL equations. This can be done by first showing that any KCL equation can be expressed as a linear combination of nodal KCL equations.

Proof (Induction): The demonstration will be made by induction. We first suggest a way to construct an arbitrary connected circuit. We start with an arbitrary branch of the circuit and build the circuit by adding one branch at a time so that at any time the circuit is connected. Then, we show that the theorem holds at the beginning of the construction procedure. We complete the proof by showing that if the theorem holds before any step in our procedure, then it remains valid afterward.

In the proof we will use $N_1,\cdots,N_n$ to denote the nodes of the circuit, $B_1,\cdots,B_b$ to denote the branches, i_k and v_k, respectively, for the current and voltage of branch B_k.

It is obvious that the theorem holds for a circuit consisting of only one initial branch. For this case, $n=2$ and $b=1$, and the theorem implies that there is one independent KCL equation and no independent KVL equations. This is indeed the case because we can write one KCL equation $i_1=0$, but no meaningful KVL equation because there are no loops.

Now we prove that if the theorem holds for the existing circuit at a specific stage in the construction procedure, the addition of a new branch does not invalidate it. We assume that at a certain stage of the construction the circuit consists of n nodes and b branches. The theorem predicts that there are $n-1$ independent KCL equations and $b-n+1$ independent KVL equations.

There are two possible ways to add a branch to the already connected circuit. In the first case the new branch

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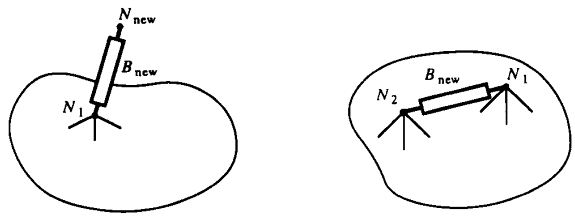


Case 1 Case 2

Fig. 1. Addition of one branch.

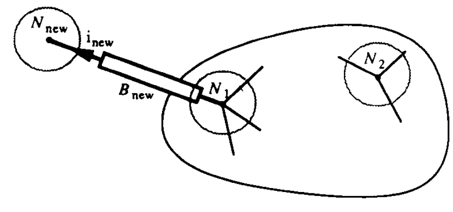


Fig. 2. The modified nodal KCL's.

has only one node in common with the nodes already in the circuit and in the second case both ends of the new branch connect to nodes already in the circuit (see Fig. 1). The two cases will be analyzed separately.

Case 1: In the case of adding a branch B_{new} having only one node N_1 in common with nodes already in the connected circuit, both the number of nodes and branches in the connected circuit increase by one. In order to show that the theorem remains valid in this case we must show that the number of independent KVL equations is $(b+1)-(n+1)+1=b-n+1$; in other words, unchanged, and the number of independent KCL equations, $(n+1)-1=n$, increases by 1.

The same set of KVL equations that existed before the addition of the branch B_{new} will apply to the augmented circuit as well. No loop that existed in the circuit before is affected by the addition of the new branch and no new loop is formed. Therefore, the number of independent KVL equations doesn't change.

The only thing left to be shown for this case is the number of independent KCL equations. By the induction hypothesis, before adding the new branch, we could write $n-1$ independent KCL equations at nodes. The addition of the new branch adds a new equation for node N_{new}, modifies the KCL equation for node N_1, and leaves the rest of the KCL equations at nodes unchanged, as shown in Fig. 2. The modified equation for node N_1 will be

$$\sum_{B_k \in \mathcal{N}_1} i_k - i_{\text{new}} = 0 \tag{1}$$

where $\mathcal{N}_1$ is used to denote the set of branches incident to node N_1 before the addition of the new branch. Observe that the KCL equation at node N_{new} is $i_{\text{new}}=0$, and therefore, the equation at node N_1 is in fact unchanged. We can conclude, therefore, that the system of KCL equations at nodes for the extended circuit consists of the set of old KCL equations with the addition of $i_{\text{new}}=0$. This equation is independent of the old set of KCL equations because, having a variable i_{new} which doesn't appear in any other equation, it cannot be obtained from them. Therefore, the number of linearly independent KCL equations is increased by 1. We have, therefore, shown that for Case 1, the number of independent Kirchhoff equations remains as predicted by the theorem.

Case 2: In the case of adding a branch B_{new} between the nodes N_1 and N_2 already in the connected circuit, the number of branches increases by 1 and the number of nodes remains constant. To show that the theorem remains valid in this case too, we have to show that the number of independent KCL equations remains constant and the number of linearly independent KVL equations, $(b+1)-n+1=b-n+2$, increases by 1.

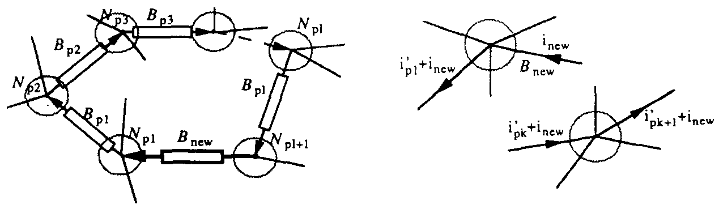
Again we consider the KCL equations at nodes written for the unaugmented circuit. By hypothesis there are $n-1$ independent equations. The addition of the branch B_{new} from node N_1 to node N_2 will modify only the two KCL equations at these nodes. Since the total number of KCL equations at nodes will not change, the only thing that we have to show is that the number of linearly independent ones remains the same.

Consider a loop formed by the new branch with old branches in the augmented circuit, as shown in Fig. 3(a). Such a loop is ensured to exist because the initial circuit, due to our construction method, is connected, so at least one path $B_{p1}, B_{p2}, \cdots, B_{pl}$ exists between nodes N_1 and N_2. This path, together with the new branch will form a loop in the augmented circuit. We can assume without loss of generality that the current directions are as shown in Fig. 3(a). In the new system of KCL equations at nodes we can decompose the currents i_{pk}, $k=1,\cdots,l$ as follows:

$$\begin{aligned} i_{p1} &= i'_{p1} + i_{\text{new}} \\ i_{p2} &= i'_{p2} + i_{\text{new}} \\ &\cdots \\ i_{pl} &= i'_{pl} + i_{\text{new}}. \end{aligned} \tag{2}$$

All the nodes in the loop have a loop branch entering the node and one leaving it. If the nodal KCL equations of the augmented circuit are written down the variable i_{new} simply cancels; see Fig. 3(b). The resulting system of equations will be identical to the system for the unaugmented circuit, where only the variables $i_{p1}, i_{p2}, \cdots, i_{pl}$ are replaced by $i'_{p1}, i'_{p2}, \cdots, i'_{pl}$. The number of independent KCL equations in this system evidently doesn't change.

The only thing left to be shown is that only one independent KVL equation is added to the original circuit with the addition of the new branch B_{new}. All the loops



(a) (b)

Fig. . The loop formed by the new branch.

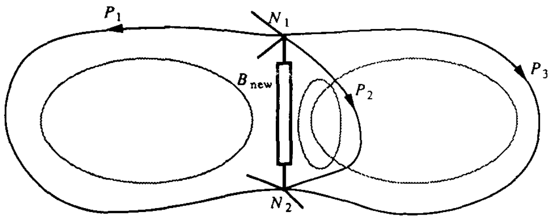


Fig. 4. Loops formed by the added branch with old paths.

existing in the original circuit will be present in the augmented circuit, too, and the KVL equations corresponding to them will remain valid. Assume that the paths $\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_p$ existed in the original circuit from the node N_1 to N_2. Each of these paths will form a new loop with the added branch B_{new} as shown in Fig. 4. To each of these loops will correspond a new KVL equation. We will show that only one of them is independent of the previous set. The following are all the new KVL equations that can be written for the augmented circuit.

$$\sum_{B_k \in \mathcal{P}_1} v_k + v_{new} = 0$$
$$\sum_{B_k \in \mathcal{P}_2} v_k + v_{new} = 0$$
$$\cdots$$
$$\sum_{B_k \in \mathcal{P}_p} v_k + v_{new} = 0. \qquad (3)$$

By subtracting the first equation from each of the rest of KVL equations corresponding to these loops, we get the equivalent system

$$\sum_{B_k \in \mathcal{P}_1} v_k + v_{new} = 0$$
$$\sum_{B_k \in \mathcal{P}_2} v_k - \sum_{k \in \mathcal{P}_1} v_k = 0$$
$$\cdots$$
$$\sum_{B_k \in \mathcal{P}_p} v_k - \sum_{k \in \mathcal{P}_1} v_k = 0. \qquad (4)$$

The first equation in system (4) is independent of the KVL equations of the original circuit because it contains the variable v_{new}. All the rest are actually the old KVL equations of the loops defined by paths $\mathcal{P}_1$ with $\mathcal{P}_2, \mathcal{P}_1$ with $\mathcal{P}_3$, etc., which are loops of the original circuit. We have shown that the addition of a branch in this case adds exactly one new independent KVL equation and have therefore concluded the proof of the theorem.

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