MIMO TRANSMIT BEAMPATTERN MATCHING UNDER WAVEFORM CONSTRAINTS

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ABSTRACT

In this paper, the multiple-input multiple-output (MIMO) transmit beampattern matching problem is considered. The problem is formulated to approximate a desired transmit beampattern (i.e., an energy distribution in space and frequency) and to minimize the cross-correlation of signals reflected back to the array by considering different practical waveform constraints at the same time. Due to the nonconvexity of the objective function and the waveform constraints, the optimization problem is highly nonconvex. An efficient one-step method is proposed to solve this problem based on the majorization-minimization (MM) method. The performance of the proposed algorithms compared to the state-of-art algorithms is shown through numerical simulations.

Index Terms— MIMO, waveform diversity, beampattern design, waveform constraints, nonconvex optimization.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) systems [1] have the capacity to transmit independent probing signal or waveforms from each transmit antenna. Such waveform diversity feature leads to many desirable properties for MIMO systems. For example, a modern MIMO radar has many appealing features, like higher spatial resolution, superior moving target detection and better parameter identifiability, compared to the classical phased-array radar [2, 3, 4].

The MIMO transmit beampattern matching problem is critically important in many fields, like in defense systems, communication systems, and biomedical applications. This problem is concerned with designing the probing waveforms to approximate a desired antenna array transmit beampattern (i.e., an energy distribution in space and frequency) and also to minimize the the cross-correlation of the signals reflected back from various targets of interest by considering some practical waveform constraints. The MIMO transmit beampattern matching problem appears to be difficult from an optimization point of view because the existence of the fourth-order nonconvex objective function and the possibly nonconvex waveform constraints which are used to represent desirable properties and/or enforced from an hardware implementation perspective [5].

In [6], the MIMO transmit beampattern matching problem was formulated to minimize the difference between the designed beampattern and the desired one. The formulation in [6] was modified in [7, 8] by introducing the cross-correlation between the signals. And in [8], the authors proposed to design the waveform covariance matrix to match the desired beampattern through semidefinite programming. A closed-form waveform covariance matrix design method was also proposed based on discrete Fourier transform (DFT) coefficients and Toeplitz matrices in [9, 10]. But such kind of methods can perform badly for small number of antennas. After the waveform covariance matrix is obtained, other methods should be applied to synthesize a desired waveform from its covariance matrix. For example, a cyclic algorithm was proposed in [11] to synthesize a constant modulus waveform from its covariance matrix. These methods are usually called two-steps methods. In practice, they could become inefficient and suboptimal if more waveform constraints are considered.

In [12], it was found that directly designing the waveform to match the desired beampattern can give a better performance, which is referred to as the one-step method. But the method in [12] is tailored to the constant modulus constraint and can be slow in convergence. In [13], the problem was solved based on the alternating direction method of multipliers (ADMM) [14]. However, again the proposed algorithm is only designed for dealing with unimodulus constraint.

The majorization-minimization (MM) method [15, 16] has shown its great efficiency in deriving fast and convergent algorithms to solve nonconvex problems in many different applications [17, 18]. In this paper, we propose a one-step method to directly solve the MIMO transmit beampattern matching problem based on the MM method by considering different waveform constraints. The performance of our algorithms compared to the existing algorithms is verified through numerical simulations.

2. MIMO TRANSMIT BEAMPATTERN MATCHING PROBLEM FORMULATION

A colocated MIMO radar [19] with $M$ transmit antennas in a uniform linear array (ULA), as shown in Fig. 1, is considered. Each transmit antenna can emit a different waveform $x_m(n)$ with $m = 1, 2, \ldots, M$, $n = 1, 2, \ldots, N$, where $N$ is the number of samples. Let $\mathbf{x}(n) = [x_1(n), x_2(n), \ldots, x_M(n)]^T$ be the $n$th sample of the $M$ transmit waveforms and $\mathbf{x} =$
The beampattern matching problem is formulated as follows:

\[
\begin{align*}
\text{minimize} & \quad f(\alpha, x) = J(\alpha, x) + \omega_{cc} E(x), \\
\text{subject to} & \quad x \in \mathcal{X} = \mathcal{X}_0 \cap \{ x \mid x(l) \geq c_d \} \\
& \quad \|x\|^2 = c_r^2,
\end{align*}
\]

where \(\omega_{cc}\) controls the sidelobe term, \(\mathcal{X}\) generally denotes the waveform constraint, and \(\mathcal{X}_0 = \{ x \in \mathbb{C}^{MN} \mid \|x\|^2 = c_r^2 \}\) representing the total transmit energy (power) constraint. We are also interested in other practical waveform constraints. i) **Constant modulus constraint** is to prevent the non-linearity distortion of the power amplifier to maximize the efficiency of the transmitter, which is given by

\[
\mathcal{X}_1 = \{ x \mid |x(l)| = c_d = \frac{c_r}{\sqrt{MN}} \}
\]

for \(l = 1, \ldots, MN\). ii) **Peak-to-Average Ratio (PAR) constraint** is the ratio of the peak signal power to its average power (PAR \(x = \max|x(l)|^2\) with \(1 \leq \text{PAR}(x) \leq MN\)). The PAR \((x)\) is constrained to a small threshold, so that the analog-to-digital and digital-to-analog converters can have lower dynamic range, and fewer linear power amplifiers are needed. Since \(\mathcal{X}_0\), the PAR constraint is \(\mathcal{X}_2 = \{ x \mid |x(l)| \leq c_p, \frac{c_p}{\sqrt{MN}} \leq c_r \leq c_e \}\) for \(l = 1, \ldots, MN\). iii) **Similarity constraint** is to allow the designed waveforms to lie in the neighborhood of a reference one which already can attain a good performance [20], which is denoted as \(\mathcal{X}_3 = \{ x \mid |x - x_{ref}| \leq c_r, 0 \leq c_r \leq \frac{p}{\sqrt{MN}} \}\).

Problem (3) is a constrained nonconvex problem due to the nonconvex objective and constraints. We are trying to solve it by using efficient nonconvex optimization methods.

### 3. Problem Solving Via the MM Method

#### 3.1. The Majorization-Minimization (MM) Method

The MM method [15, 21, 16] is a generalization of the well-known EM method. For an optimization problem given by

\[
\begin{align*}
\text{minimize} & \quad f(x) \quad \text{subject to} \quad x \in \mathcal{X}, \quad x \in \mathcal{X}, \\
\end{align*}
\]

instead of dealing with this problem directly which could be difficult, the MM-based algorithm solves a series of simpler subproblems with surrogate functions that majorize \(f(x)\) over \(\mathcal{X}\). More specifically, starting from an initial point \(x^{(0)}\), it produces a sequence \(\{x^{(k)}\}\) by the following update rule:

\[
x^{(k)} \in \arg \min_{x \in \mathcal{X}} \tilde{f}(x, x^{(k-1)}),
\]

where the surrogate majorizing function \(\tilde{f}(x, x^{(k)})\) satisfies

\[
\begin{align*}
\tilde{f}(x^{(k)}, x^{(k)}) &= f(x^{(k)}), \quad \forall x^{(k)} \in \mathcal{X}, \\
\tilde{f}(x, x^{(k)}) &\geq f(x), \quad \forall x, x^{(k)} \in \mathcal{X}, \\
\tilde{f}(x^{(k)}, x^{(k)}; d) &= f'(x^{(k)}; d), \quad \forall d, \text{ s.t. } x^{(k)} + d \in \mathcal{X}.
\end{align*}
\]

The objective function value is monotonically nonincreasing at each iteration. To use the MM method, the key step is to find a majorizing function to make the subproblem easy to solve, which will be discussed in the following subsections.

#### 3.2. Majorization Steps For The Beampattern Matching Term \(J(\alpha, x)\)

In this section, we discuss the majorization steps, i.e., how to construct a good majorizing function for the beampattern matching term \(J(\alpha, x)\) in (1). First, we have
\[ J(\alpha, x) = \sum_{\theta \in \Theta} \omega(\theta) |\alpha P(\theta, x) - P(\theta, x)|^2 \]  
\[ = \alpha^2 \sum_{\theta \in \Theta} \omega(\theta) \frac{p(\theta)}{P(\theta, x)}^2 \]  
\[ + \sum_{\theta \in \Theta} \omega(\theta) \left( \frac{P(\theta, x)}{P(\theta, x)}^2 \right) , \]

which is a quadratic function in variable \( \alpha \). Then, it follows that the minimum of \( J(\alpha, x) \) is attained when

\[ \alpha(x) = \sum_{\theta \in \Theta} \omega(\theta) p(\theta) P(\theta, x) / \sum_{\theta \in \Theta} \omega(\theta) p(\theta) . \]

Substituting \( \alpha(x) \) back into \( J(\alpha, x) \) and considering \( P(\theta, x) = \text{Tr}(xx^H \mathbf{A}(\theta)) = \text{vec}(xx^H)^H \text{vec}(\mathbf{A}(\theta)) \), we get

\[ J(x) = \sum_{\theta \in \Theta} \omega(\theta) \left( \text{vec}(xx^H)^H \text{vec}(\mathbf{A}(\theta)) \right)^2 \]

\[ - \left( \sum_{\theta \in \Theta} \omega(\theta) p(\theta) \right)^{-1} \left( \text{vec}(xx^H)^H \text{vec}(\sum_{\theta \in \Theta} \omega(\theta) x^T) \right)^2 = \text{vec}(xx^H)^H \mathbf{H}_J \text{vec}(xx^H) , \]

where

\[ \mathbf{H}_J = \sum_{\theta \in \Theta} \omega(\theta) \text{vec}(\mathbf{A}(\theta)) \text{vec}(\mathbf{A}(\theta))^H - \left( \sum_{\theta \in \Theta} \omega(\theta) p(\theta) \right)^{-1} \text{vec}(\sum_{\theta \in \Theta} \omega(\theta) \text{vec}(\mathbf{A}(\theta)) \text{vec}(\sum_{\theta \in \Theta} \omega(\theta) x^T) \right)^H , \]

and it is easy to see that \( J(x) \) is a quartic function in \( x \). Next, we introduce a useful lemma.

**Lemma 1.** Let \( \mathbf{A} \in \mathbb{H}^{K \times K} \) and \( \mathbf{B} \in \mathbb{H}^{K \times K} \) such that \( \mathbf{B} \succeq \mathbf{A} \). At any point \( x_0 \in C^K \), the quadratic function \( x^T \mathbf{A} x \) is majorized by \( x^T \mathbf{B} x + 2\text{Re}(x^T (\mathbf{B} - \mathbf{A}) x_0) + x_0^T (\mathbf{B} - \mathbf{A}) x_0 \).

**Proof.** Notice that \( (x - x_0)^T (\mathbf{B} - \mathbf{A}) (x - x_0) \geq 0 \).

Based on Lemma 1, we can choose \( \psi_{j,1} \geq \lambda_{\max}(\mathbf{H}_J) \), and because \( \psi_{j,1} I \succeq \mathbf{H}_J \), at iterate \( x^{(t)} \) we have

\[ J(x) \leq \psi_{j,1} \text{vec}(xx^H)^H x \]

\[ + 2\text{Re} \left( \text{vec}(xx^H)^H (\mathbf{H}_J - \psi_{j,1} I) \text{vec}(x^{(t)}x^{(t)H}) \right) \]

\[ + \text{vec}(x^{(t)}x^{(t)H})^H (\psi_{j,1} I - \mathbf{H}_J) \text{vec}(x^{(t)}x^{(t)H}) \],

where \( \text{vec}(xx^H)^H \text{vec}(xx^H) = ||x||_2^4 = c_2^4 \), the first term is just a constant. Then after ignoring the constant terms, we get the following majorizing function for \( J(x) \):

\[ \mathcal{J}_1(x, x^{(t)}) \approx 2x^T (\mathbf{M}_J - \psi_{j,1}x^{(t)}x^{(t)H}) x \]

\[ \simeq 2 \text{Re} \left( \text{vec}(xx^H)^H (\mathbf{H}_J - \psi_{j,1} I) \text{vec}(x^{(t)}x^{(t)H}) \right) , \]

where “\( \simeq \)” stands for “equivalence” up to additive constants. Substituting \( \mathbf{H}_J \) back into function \( \mathcal{J}_1(x, x^{(t)}) \) and dropping the constants, we have

\[ \mathcal{J}_1(x, x^{(t)}) \approx 2x^T (\mathbf{M}_J - \psi_{j,1}x^{(t)}x^{(t)H}) x \]

\[ \simeq 2x^T (\mathbf{M}_J - \psi_{j,1}x^{(t)}x^{(t)H}) x \]

where \( \mathbf{M}_J = \sum_{\theta \in \Theta} \omega(\theta) ( P(\theta, x^{(t)}) - \alpha(x) x^{(t)} ) \mathbf{A}(\theta) \). It is easy to see that after majorization, the majorizing function \( \mathcal{J}_1(x, x^{(t)}) \) becomes quadratic in \( x \) rather than quartic in \( J(x) \). However, using this function as the objective to solve is still hard due to the wavefront constraint \( \lambda_2^2 \). So we propose to majorize \( \mathcal{J}_1(x, x^{(t)}) \) again to simplify the problem to solve in each iteration. Thus, we can consider choosing \( \psi_{j,2} \geq \lambda_{\max}(\mathbf{M}_J) \geq \lambda_{\max}(\mathbf{M}_J - \psi_{j,1}x^{(t)}x^{(t)H}) \) for majorization, where we can have the following useful property.

\[ \text{vec}(xx^H)^H \mathbf{H}_E \text{vec}(xx^H) , \]

where \( \mathbf{H}_E = \sum_{\theta \in \Theta, i \neq j} \omega_{i,j} \text{vec}(\mathbf{A}(\theta_i, \theta_j)) \text{vec}(\mathbf{A}(\theta_i, \theta_j))^H \). Then, based on Lemma 1, by choosing \( \psi_{E,1} \geq \lambda_{\max}(\mathbf{H}_E) \) and \( \psi_{E,2} \geq \lambda_{\max}(\mathbf{M}_E - \psi_{E,1}x^{(t)}x^{(t)H}) \), we can get the majorizing functions at iterate \( x^{(t)} \) written as follows:

\[ \mathcal{E}_1(x, x^{(t)}) \approx 2x^T (\mathbf{M}_E - \psi_{E,1}x^{(t)}x^{(t)H}) x \]

\[ \leq \mathcal{E}_2(x, x^{(t)}) \approx -4x^T y_E , \]

where \( \mathbf{M}_E = \sum_{\theta \in \Theta, i \neq j} \omega_{i,j} P_{cc}(\theta_i, \theta_j, x) \mathbf{A}(\theta_i, \theta_j) \) and \( y_E = -(\mathbf{M}_E - c_2^2 \psi_{E,1} \mathbf{I} - \psi_{E,2} I) x^{(t)} \).

**3.4. Solving The Majorized Subproblem in MM**

By combing the two majorizing functions \( \mathcal{J}_2(x, x^{(t)}) \) and \( \mathcal{E}_2(x, x^{(t)}) \), the overall majorizing function at iterate \( x^{(k)} \) for the objective \( f(x) \) is given as follows:

\[ f(x) \leq \mathcal{J}(x, x^{(t)}) \approx \mathcal{J}_2(x, x^{(t)}) + \omega_{cc} \mathcal{E}_2(x, x^{(t)}) \]

\[ \approx -4x^T y_E - 4\omega_{cc} \text{Re}(x^H y_E) = -\text{Re}(x^H y_E) , \]

which is Hermitian Toeplitz, \( \mathbf{F} \) as a \( 2M \times 2M FFT \) matrix, and \( b = [b_0, b_1, \ldots, b_{M-1}, 0, b_{M-1}, \ldots, b_1]^T \). Then, we have \( \mathbf{M}_J = \mathbf{I}_N \otimes \mathbf{B} \), \( \lambda_{\max}(\mathbf{M}_J) = \lambda_{\max}(\mathbf{B}) \), and \( \lambda_{\max}(\mathbf{B}) \leq \lambda_{\mu} = \frac{1}{2} \left( \frac{\mu}{1 \leq i \leq M} \right) \), where \( \mu = \text{FFT transform of \( \mathbf{B} \).} \)
where $y = -4 \left( M_J + \omega_{cc} M_E - c_n^2 (\psi_{J,1} + \omega_{cc} \psi_{E,1}) I - (\psi_{J,2} + \omega_{cc} \psi_{E,2}) I \right) x(t)$.

Finally, by majorizing the objective function in (3) using the MM method, the subproblem we need to solve at each iteration is given as follows:

$$\min_{x} \quad \mathcal{L}(x, x^{(t)}) \simeq -\Re (x^H y) \quad \text{s.t.} \quad x \in X.$$  \hspace{1cm} (7)

For problem (7), as to different interested waveform constraints, closed-form optimal solutions $x^*$ can be derived, which are summarized in the following lemma.

**Lemma 3.** i) For fixed energy constraint (i.e., $X = X_0$), $x^* = c^* y / \| y \|_2^2$; ii) for constant modulus constraint (i.e., $X = X_1$), $x^* = c^* e^{i \arg(y)^3}$; iii) for fixed energy with PAR constraint (i.e., $X = X_0 \cap X_2$), the solution $x^*$ can be found in [24, Alg. 2]; iv) for constant modulus with similarity constraint (i.e., $X = X_1 \cap X_3$), the solution $x^*$ can be found in [25].

### 3.5. The MM-Based Beampattern Matching Algorithm

Based on the MM method, in order to solve the original problem (3), we just need to iteratively solve the subproblem (7) with a closed-form solution update in Lemma 3 at each iteration. The overall algorithm is summarized as follows.

**Input:** $a(\theta), p(\theta), x^{(0)}$ and $t = 0$.

**Repeat**

1. Compute $M_J, M_E, \psi_{J,1}, \psi_{E,1}, \psi_{J,2}, \psi_{E,2}$ and $y$;
2. Update $x^{(t)}$ in a closed-form according to Lemma 3;
3. $t = t + 1$;

**Until** $x$ and $f(x)$ satisfy a termination criterion.

**Output:** $\alpha, x$.

### 4. NUMERICAL SIMULATIONS

The performance of the proposed algorithm for MIMO transmit beampattern matching is evaluated by numerical simulations. A colocated MIMO radar system is considered with a ULA comprising $M = 10$ antennas with half-wavelength spacing between adjacent antennas. Without loss of generality, the total transmit power is set to $c_n^2 = 1$. Each transmit pulse has $N = 32$ samples. The range of angle is $\Theta = (-90^\circ, 90^\circ)$ with spacing $1^\circ$ under which the weight $\omega(\theta) = 1$ for $\theta \in \Theta$, and $\omega_{cc} = 0$, which is the same setting as [13]. We consider a desired beampattern with three targets or mainlobes ($K = 3$) at $\theta_1 = -40^\circ$, $\theta_2 = 0^\circ$, $\theta_3 = 40^\circ$, and each width of them is $\Delta \theta = 20^\circ$. The desired beampattern is

$$p(\theta) = \begin{cases} 1, & \theta \in [\theta_k - \Delta \theta/2, \theta_k + \Delta \theta/2], \ k = 1, 2, K \\ 0, & \text{otherwise.} \end{cases}$$

We compare the convergence property over iterations of the objective function for the beampattern matching problem under unimodulus waveform constraint by using the proposed MM-based algorithm (denoted as MM-based algorithm (prop.)) and the ADMM-based algorithm in [13] (denoted as ADMM-based algorithm), which is shown in Fig. 2.

As shown in Fig. 2, the MM-based algorithm can have a monotonic convergence property. And it can converge within 20 iterations which is faster than the benchmark algorithm.

Then, we also compare the matching performance of the designed beampatterns in terms of the mean-squared error (MSE) defined as

$$\text{MSE} (P(\theta, x)) = E \left[ \sum_{\theta \in \Theta} \omega(\theta) |\alpha p(\theta) - P(\theta, x)|^2 \right].$$

In Fig. 3, we show the simulation results for MSE ($P(\theta, x)$) by using different design methods.

### 5. CONCLUSIONS

This paper has considered the MIMO transmit beampattern matching problem. Efficient algorithms have been proposed based on the MM method. Numerical simulations show that the proposed algorithms are efficient in solving the beampattern matching problem and can obtain a better performance compared to the the state-of-art method.
6. REFERENCES


